Universality classes and fluctuations in disordered systems

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Waves transmitted through disordered media show increasing fluctuations with thickness of material so that averages of different properties of the wavefield have very different scaling with thickness traversed. We have been able to classify these properties according to a scheme that is independent of the nature of the medium, such that members of a class have a universal scaling independent of the nature of the medium. We apply this result to trace$(T_L T_L^{\dagger})^M$, where T_L is the amplitude transmission matrix. The eigenfunctions of $T_L T_L^{\dagger}$ define a set of channels through which the current flows, and the eigenvalues are the corresponding transmission coefficients. We prove that these coefficients are either ≈ 0 or ≈ 1. As L increases more channels are shut down. This is the *maximal fluctuation theorem*: fluctuations cannot be greater than this. We expect that our classification scheme will prove of further value in proving theorems about limiting distributions. We show by numerical simulations that our theorem holds good for a wide variety of systems, in one, two and three dimensions.

1. Introduction

Disordered media scatter waves incident upon them, and induce in the scattered wavefield a degree of disorder which is far more extreme than the physical disorder of the medium itself. The fluctuations are brought about by multiple scattering of the wavefield and its ability to interfere constructively or destructively with itself. The most extreme instances are found in the absence of absorption, when multiple scattering can have free rein.

As a specific example we consider the system shown in figure 1 in which waves are incident on a slab of disordered material of finite thickness in one direction, but effectively infinite in the other directions. We shall assume that the slab is statistically homogeneous. The waves can be either transmitted through or reflected from the slab. Statistics in the transmission coefficient pose particular challenges: as the thickness, L, of the slab is increased fluctuations in the transmission coefficient, far from settling down to some 'average' value, become more extreme. The question is what can we say about the limiting scattering properties of a thick slab of material? As far as we are aware this question has only been addressed for special cases. The results we derive here apply in very general circumstances.

With increasing thickness, L, physically observable quantities typically scale exponentially with L, but other limiting behaviour, such as a power law, is possible. Sometimes physically distinct quantities can have the same asymptotic behaviour: we first proved this for one-dimensional systems, but later observed from numerical simulations that our results seemed to have a wider validity. Finally we were able to

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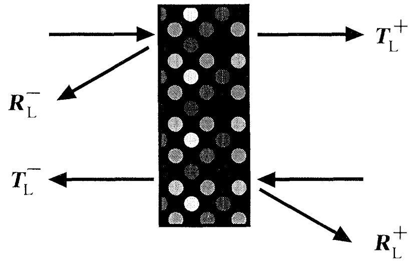


Figure 1. A slab of medium, thickness L, scatters waves.

Figure 2

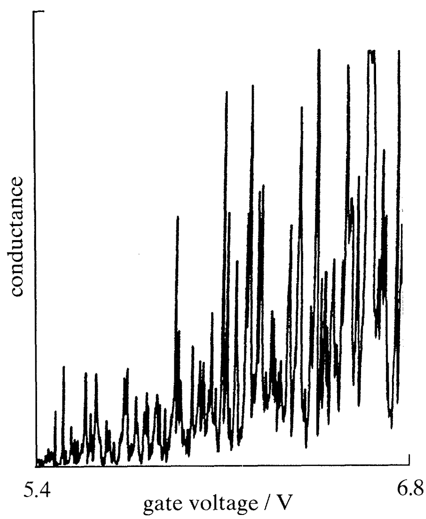


Figure 3

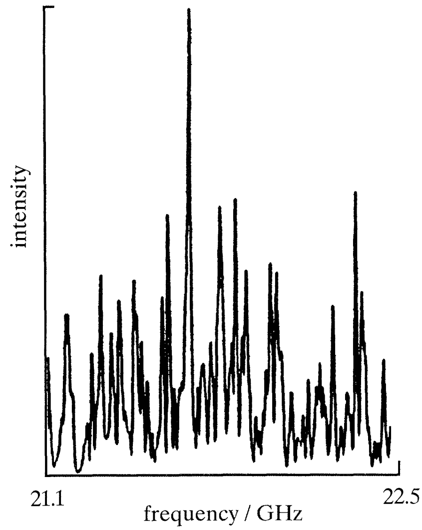


Figure 2. Conductance of a silicon metal oxide field effect transistor measured at 50 mK.

Figure 3. Spectral intensity fluctuations for microwaves passing through a 140 cm length of tube filled with half-inch diameter polystyrene balls.

prove that in many cases this coincidence of the asymptotic behaviour follows from general considerations of the mathematical structure, and does not depend on the degree or nature of the disorder, nor even on the dimensionality. Hence our title universality classes: quantities are placed in a class according to their asymptotic scaling, and our claim is that the classification is universal, independent of degree of disorder, and of dimensionality.

The problem we address has many specific realizations. Perhaps the most well known is that of electrons in disordered semiconductors. At low temperatures electron energy loss due to phonon scattering is greatly reduced, and extreme fluctuations in conductance have been observed, see figure 2. However many other instances are known, such as the case of transmission of microwaves through a tube packed with a random array of disordered dielectric spheres, see figure 3. In fact any experiment that has waves interacting with disorder must address the question of fluctuations: they will always be intense and will usually affect the experiment in some crucial aspect. A better understanding of these fundamental physical systems is needed and we hope that this paper will prove a useful step in that direction.

Statistical distributions are characterized by their moments. For example in a one-dimensional system the distribution of transmitted intensities, $P(|T|^2)$, is specified by

$\langle|T|^{2N}\rangle$. Therefore averages of various quantities, particularly of powers of those quantities, play a central role in the statistics. We shall use $M(L)$ to denote an average quantity for a system of length L. We shall prove that the Ms can be assigned to one of a set of universality classes. The number of possible classes is infinite. Label the pth member of the Sth universality class as M_{pS}. Then our result states that averages that are members of the same universality class have the same asymptotic dependence on the length, L, of the system:

$$\lim_{L\to\infty} M_{pS}/M_{p'S} = C^S_{pp'}, \tag{1}$$

where $C^S_{pp'}$ is a constant independent of L.

A specific example of the theorem is given by a one-dimensional system. In one-dimensional systems the reflection coefficient and its moments, $\langle|R|^{2N}\rangle$, all belong to the same universality class. Using the relationship,

$$|T|^2+|R|^2 = 1, \tag{2}$$

which holds in the absence of absorption, the implication is that all the moments of $|T|^2$ belong to the same universality class. In fact in an earlier paper (Kirkman & Pendry 1984*a*) explicit values were calculated for the ratios,

$$\begin{aligned}\lim_{L\to\infty} \langle|T_L|^{2N}\rangle/\langle|T_L|^2\rangle &= C_N \\ &= \Gamma^2(N-\tfrac{1}{2})\,\Gamma^2(1)/\Gamma^2(N)\,\Gamma^2(\tfrac{1}{2}),\end{aligned} \tag{3}$$

in agreement with the general theorem which we shall prove below. This result is a surprising one because $\langle|T_L|^2\rangle$ decreases exponentially with L. It has some remarkable implications for the distribution $P(|T_L|^2)$: suppose for a moment that the constants $C_N = 1$. This would imply that the distribution was completely bimodal: only $|T_L|^2 = 1$ and $|T_L|^2 = 0$ are consistent with this result. Compared with $\exp(-L)$, it is certainly true that $C_N \approx 1$ so that either,

$$|T_L|^2 \approx 1, \tag{4}$$

or,

$$|T_L|^2 \approx 0. \tag{5}$$

As L increases the exponential decrease of $\langle|T_L|^2\rangle$ comes about by the sharply reduced incidence of the $|T_L|^2 \approx 1$ specimens.

In §2 we define the transfer matrix, the main tool of our proof, and show how it can be generalized by taking direct products which can then be factored according to the irreducible representations of the symmetric group. In §3 we give the proof of our theorem, and discuss the difficulties presented when the generalized transfer matrices have continuous spectra. In §4 we give numerical demonstrations of the theorem in one, two, and three dimensions. Section 5 examines an unusual case where translational symmetry of the *distribution function* of disorder in the medium defines a universality class.

2. The generalized transfer matrix

Our approach is based on transfer matrix theory in conjunction with the application of group theory. The methodology can be summarized as follows: our slab of disordered material is notionally divided into statistically independent slices, see figure 4. For the nth slice we have transmission and reflection coefficients t_n and

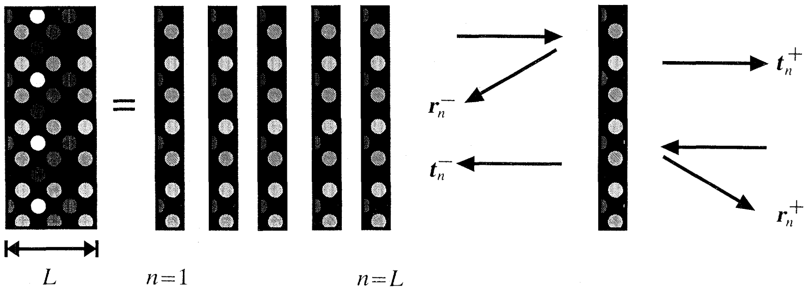


Figure 4. Decomposition of a slab into L slices. Scattering from a thin slice is easy to calculate.

r_n which in two and three dimensions will be matrices. The whole slab has transmission and reflection coefficients T_L and R_L. For each of these slices we define a *transfer matrix*,

$$\boldsymbol{X}_n = \begin{pmatrix} (\boldsymbol{t}_n^-)^{-1} & -(\boldsymbol{t}_n^-)^{-1}\boldsymbol{r}_n^- \\ \boldsymbol{r}_n^+(\boldsymbol{t}_n^-)^{-1} & \boldsymbol{t}_n^+ - \boldsymbol{r}_n^+(\boldsymbol{t}_n^-)^{-1}\boldsymbol{r}_n^- \end{pmatrix}, \tag{6}$$

which has the property that,

$$\boldsymbol{X}_L(\boldsymbol{T}_L^+, \boldsymbol{T}_L^-, \boldsymbol{R}_L^+, \boldsymbol{R}_L^-) = \prod_{n=1}^{L} \boldsymbol{X}_n(\boldsymbol{t}_n^+, \boldsymbol{t}_n^-, \boldsymbol{r}_n^+, \boldsymbol{r}_n^-), \tag{7}$$

where

$$\boldsymbol{X}_L = \begin{pmatrix} (\boldsymbol{T}_L^-)^{-1} & -(\boldsymbol{T}_L^-)^{-1}\boldsymbol{R}_L^- \\ \boldsymbol{R}_L^+(\boldsymbol{T}_L^-)^{-1} & \boldsymbol{T}_L^+ - \boldsymbol{R}_L^+(\boldsymbol{T}_L^-)^{-1}\boldsymbol{R}_L^- \end{pmatrix}. \tag{8}$$

Notice that $\boldsymbol{X}_L$ has the same functional dependence on the scattering properties of the slab, as $\boldsymbol{X}_n$ does on the scattering properties of the nth slice. Equation (7) we shall refer to as the *fundamental theorem*. Given the scattering properties of individual slices it offers an elegant and compact expression for the scattering properties of the slab of L slices.

In a random system equation (7) also offers the possibility of taking averages because, provided only that the slices are statistically independent,

$$\langle \boldsymbol{X}_L \rangle = \prod_{n=1}^{L} \langle \boldsymbol{X}_n \rangle, \tag{9}$$

which gives the average of any element of $\boldsymbol{X}$. The taking of proper averages is one of the most complex issues in statistical mechanics, yet here we have an elegant and straightforward prescription. It is to the fundamental theorem and its ability to solve the averaging problem that our results owe their generality.

Unfortunately many of the averages which interest us do not appear as elements of $\boldsymbol{X}$. It was to circumvent this difficulty that we generalized the transfer matrix in a series of papers. We begin by constructing the direct product of $\boldsymbol{X}$ with itself,

$$\boldsymbol{X}^{\otimes 2} = \boldsymbol{X} \otimes \boldsymbol{X}, \tag{10}$$

which consists of all the pairwise products of the elements of $\boldsymbol{X}$. In general,

$$\boldsymbol{X}^{\otimes N} = \boldsymbol{X} \otimes \boldsymbol{X} \otimes \boldsymbol{X} \cdots \otimes \boldsymbol{X}. \tag{11}$$

The new matrix $\boldsymbol{X}^{\otimes N}$ contains all Nth order integer powers of elements of $\boldsymbol{X}$ and, most important of all, obeys the fundamental theorem,

$$\boldsymbol{X}_L^{\otimes N} = \prod_{n=1}^{L} \boldsymbol{X}_n^{\otimes N}. \tag{12}$$

We have shown how to generalize the transfer matrix further to non-integer, negative, or complex values of N (Kirkman & Pendry 1984*b*; Barnes & Pendry 1991).

It is not immediately obvious, but in the process of generalizing the transfer matrix we have introduced a symmetry into the problem. In equation (11) we are taking the direct product of N identical matrices, and therefore we can permute the matrices without affecting the results. This permutation symmetry implies a factorization of the product: $\boldsymbol{X}^{\otimes N}$ must factorize into irreducible subspaces in a manner dictated by the symmetric group. We can draw a physical analogy with a many particle system described by a hamiltonian. Once the symmetry of the particles under exchange has been specified, bosonic, or fermionic, we must project the hamiltonian onto the subspace of appropriate symmetry. Likewise our generalized transfer matrix can be projected onto symmetric or antisymmetric subspaces. In fact there is a wider choice of subspaces, and any symmetry described by a Young tableau (Littlewood 1950; Hammermesh 1962) can be identified with the appropriate subspace of $\boldsymbol{X}^{\otimes N}$. In our theory we go even further and talk of the permutation of N objects where N can be a complex number!

It is these symmetrized subspaces that define the universality classes for us. They are hermetically sealed compartments in the sense that the symmetrically reduced $\boldsymbol{X}_L^{\otimes N}$ is given solely in terms of the symmetrically reduced $\boldsymbol{X}_n^{\otimes N}$ for the individual slices,

$$\boldsymbol{X}_{SL}^{\otimes N} = \prod_{n=1}^{L} \boldsymbol{X}_{Sn}^{\otimes N}, \tag{13}$$

where the subscript S labels the subspace.

One further symmetry may appear in two- or three-dimensional systems. If the disorder is statistically homogeneous within each of the slices, then an equivalent of Bloch's theorem exists. To state the theorem, let us first describe the scattering by a slice in a plane wave representation, so that the transmission and reflection matrices have subscripts as follows,

$$\boldsymbol{t}_{n;\boldsymbol{kk}'}, \quad \boldsymbol{r}_{n;\boldsymbol{kk}'}, \tag{14}$$

where $\boldsymbol{k}$ labels the wavevector. If the slices were not disordered, and were translationally invariant parallel to the slice, then Bloch's theorem would be obeyed. Denote by a tilde quantities for a translationally invariant system,

$$\left.\begin{aligned} \tilde{\boldsymbol{t}}_{n;\boldsymbol{kk}'} &= \tilde{\boldsymbol{t}}_{n;\boldsymbol{kk}'}\,\delta_{\boldsymbol{kk}'}, \\ \tilde{\boldsymbol{r}}_{n;\boldsymbol{kk}'} &= \tilde{\boldsymbol{r}}_{n;\boldsymbol{kk}}\,\delta_{\boldsymbol{kk}'}, \end{aligned}\right\} \tag{15}$$

and in this case,

$$\tilde{\boldsymbol{X}}_{n;\boldsymbol{kk}'} = \tilde{\boldsymbol{X}}_{n;\boldsymbol{kk}}\,\delta_{\boldsymbol{kk}'}. \tag{16}$$

However, even if the slices are not translationally invariant, but only so on average, then the average of $\boldsymbol{X}$ will retain this property,

$$\langle \boldsymbol{X}_{n;\boldsymbol{kk}'} \rangle = \langle \boldsymbol{X}_{n;\boldsymbol{kk}} \rangle\,\delta_{\boldsymbol{kk}'}. \tag{17}$$

That is to say, the generalized $\boldsymbol{X}$s also obey a form of Bloch's theorem. Their subscripts comprise arrays of $\boldsymbol{k}$s, one $\boldsymbol{k}$ for each component of the direct product,

$$\begin{gathered} \langle \boldsymbol{X}^{\otimes N}_{\boldsymbol{k}_1\boldsymbol{k}_2\cdots\boldsymbol{k}_N;\boldsymbol{k}'_1\boldsymbol{k}'_2\cdots\boldsymbol{k}'_N} \rangle = 0, \\ \text{unless} \quad \boldsymbol{k}_1+\boldsymbol{k}_2+\cdots+\boldsymbol{k}_N = \boldsymbol{k}'_1+\boldsymbol{k}'_2+\cdots+\boldsymbol{k}'_N. \end{gathered} \tag{18}$$

Hence the generalized transfer matrices have a further factorization into subspaces labelled by the total momentum,

$$q = k_1 + k_2 + \cdots + k_N. \tag{19}$$

3. Universality classes: the general theorem

Let us suppose that the quantity $M_{pS}(L)$ can be expressed as the ijth element of a generalized transfer matrix,

$$M_{pS}(L) = (\boldsymbol{X}_{SL}^{\otimes N})_{ij}, \tag{20}$$

and that another quantity $M_{p'S}(L)$ can be expressed as the $i'j'$th element of the same matrix,

$$M_{p'S}(L) = (\boldsymbol{X}_{SL}^{\otimes N})_{i'j'}. \tag{21}$$

We wish to prove that the ratio of these quantities tends to a constant with increasing L; see equation (1).

First we make use of equation (13) to expand the quantities in terms of the eigenvectors of $\boldsymbol{X}_{Sn}^{\otimes N}$,

$$\begin{aligned} M_{pS}(L) = (\boldsymbol{X}_{SL}^{\otimes N})_{ij} &= \left(\prod_{n=1}^{L} \boldsymbol{X}_{Sn}^{\otimes N}\right)_{ij} \\ &= \sum_s e_s^L \langle i \,|\, s\rangle \langle s \,|\, j\rangle, \end{aligned} \tag{22}$$

where $|s\rangle$ and $\langle s|$ are respectively right and left eigenvectors of $\boldsymbol{X}_{Sn}^{\otimes N}$, and e_s is the corresponding eigenvalue. The nature of the eigenvalue spectrum is crucial, and varies according to which particular $\boldsymbol{X}_{Sn}^{\otimes N}$ we are concerned with. In general the matrix has infinite dimensions and therefore an infinite number of eigenvalues. In many instances the eigenvalues are constrained by unitarity to have,

$$|e_s| \leqslant 1 \tag{23}$$

and in all cases we have encountered the spectrum is bounded by a maximum modulus,

$$|e_s| \leqslant |e_{s\,\mathrm{max}}|. \tag{24}$$

The main factor differentiating the classes is whether the spectrum is discrete or continuous in the immediate vicinity of the maximum modulus. A different proof is required in each of these cases:

(a) *Case 1: eigenvalue spectrum discrete*

In this case the theorem follows in a simple and obvious manner from (22):

$$\begin{aligned} \lim_{L\to\infty} M_{pS}(L)/M_{p'S}(L) &= \lim_{L\to\infty} \frac{\Sigma_s\, e_s^L \langle i \,|\, s\rangle \langle s \,|\, j\rangle}{\Sigma_s\, e_s^L \langle i' \,|\, s\rangle \langle s \,|\, j'\rangle} \\ &= \frac{e_{s\,\mathrm{max}}^L \langle i \,|\, s\,\mathrm{max}\rangle \langle s\,\mathrm{max} \,|\, j\rangle}{e_{s\,\mathrm{max}}^L \langle i' \,|\, s\,\mathrm{max}\rangle \langle s\,\mathrm{max} \,|\, j'\rangle} \\ &= C_{pp'}^S, \end{aligned} \tag{25}$$

where,

$$C_{pp'}^S = \frac{\langle i \,|\, s\,\mathrm{max}\rangle \langle s\,\mathrm{max} \,|\, j\rangle}{\langle i' \,|\, s\,\mathrm{max}\rangle \langle s\,\mathrm{max} \,|\, j'\rangle}. \tag{26}$$

Evidently, in the case of discrete eigenvalues, the ratio approaches its asymptotic value as,

$$\lim_{L\to\infty} M_{pS}(L)/M_{p'S}(L) = C^S_{pp'} + \text{const.}\ (e_{\max-1}/e_{\max})^L. \tag{27}$$

(*b*) *Case 2: eigenvalue spectrum continuous*

As in *Case 1* we express the ratio in terms of the eigenvectors,

$$\lim_{L\to\infty} M_{pS}(L)/M_{p'S}(L) = \lim_{L\to\infty} \frac{\int \mathrm{d}e_s\, e_s^L \langle i\,|\,s\rangle\langle s\,|\,j\rangle}{\int \mathrm{d}e_s\, e_s^L \langle i'\,|\,s\rangle\langle s\,|\,j'\rangle}. \tag{28}$$

If it so happens that

$$\lim_{e_s\to e_{s\max}} \langle i\,|\,s\rangle\langle s\,|\,j\rangle = \text{const.}, \tag{29}$$

then the proof stands as in *Case 1*. In general this will not be the case. It might happen that,

$$\lim_{e_s\to e_{s\max}} \langle i\,|\,s\rangle\langle s\,|\,j\rangle = D^S_p(e_{s\max}-e_s)^{\alpha_p}, \tag{30}$$

where D^S_p is a constant, so that,

$$\lim_{L\to\infty}\int \mathrm{d}e_s\, e_s^L \langle i\,|\,s\rangle\langle s\,|\,j\rangle = \lim_{L\to\infty}\int \mathrm{d}e_s\, e_s^L D^S_p(e_{s\max}-e_s)^{\alpha_p}$$
$$= e^L_{s\max} D^S_p \Gamma(\alpha_p) L^{-\alpha_p-1}, \tag{31}$$

where Γ is the usual gamma function. In this case we obtain,

$$\lim_{L\to\infty} M_{pS}(L)/M_{p'S}(L) = \frac{e^L_{s\max} D^S_p \Gamma(\alpha_p) L^{-\alpha_p-1}}{e^L_{s\max} D^S_{p'} \Gamma(\alpha_{p'}) L^{-\alpha_{p'}-1}}$$
$$= \frac{D^S_p \Gamma(\alpha_p)}{D^S_{p'}\Gamma(\alpha_{p'})} L^{-\alpha_p+\alpha_{p'}}, \tag{32}$$

and our theorem is not obeyed. However, there is a weaker form of the theorem which is,

$$\lim_{L\to\infty} \frac{\log M_{pS}(L)}{\log M_{p'S}(L)} = \lim_{L\to\infty} \frac{L_{s\max} + \log\,(D^S_p \Gamma(\alpha_p) L^{-\alpha_p-1})}{L_{s\max} + \log\,(D^S_{p'} \Gamma(\alpha_{p'}) L^{-\alpha_{p'}-1})}$$
$$= 1. \tag{33}$$

Yet for many instances of a continuous eigenvalue spectrum the strong form of the theorem still holds. For a proof it is necessary to delve into details of the eigenvectors. Let us assume that, although the matrix $\boldsymbol{X}^{\otimes N}_{Sn}$ has infinite dimensions, the ijth element corresponds to finite i and j. Since the spectrum has been assumed continuous, the eigenvectors have components extending to infinity and their normalization must be infinite. Our proof will show that although the normalization is singular, the singularity is the same for all elements corresponding to finite i and j, so that,

$$\lim_{e_s\to e_{s\max}} \langle i\,|\,s\rangle\langle s\,|\,j\rangle/\langle i'\,|\,s\rangle\langle s\,|\,j'\rangle = C^S_{pp'}. \tag{34}$$

We start from the eigenvalue equation,

$$(\boldsymbol{X}_{Sn}^{\otimes N} - e_s \boldsymbol{I})|s\rangle = 0, \tag{35}$$

and solve for the un-normalized elements of the right eigenvector, $A_s \langle j|s\rangle$, by gaussian elimination first setting,

$$A_s \langle 1|s\rangle = 1. \tag{36}$$

Barring any accidental degeneracies this procedure will produce finite values for all finite-order elements of $A_s\langle j|s\rangle$. Similarly we can calculate the un-normalized left eigenvectors, $A_s^* \langle s|j\rangle$, and these will again be finite for all finite order elements. We calculate A_s by the normalization requirement,

$$\sum_{j=1}^{\infty} |A_s|^2 \langle j|s\rangle \langle s|j\rangle = |A_s|^2, \tag{37}$$

and, since the summation extends to infinity, A_s will in general be singular at the spectrum limit. Hence we can rewrite equation (28),

$$\lim_{L\to\infty} \frac{M_{pS}(L)}{M_{p'S}(L)} = \lim_{L\to\infty} \frac{\int \mathrm{d}e_s\, e_s^L |A_s|^{-2} [|A_s|^2 \langle i|s\rangle \langle s|j\rangle]}{\int \mathrm{d}e_s\, e_s^L |A_s|^{-2} [|A_s|^2 \langle i'|s\rangle \langle s|j'\rangle]}. \tag{38}$$

We have already proved that the factors in square brackets are non-singular for finite i and j,

$$\lim_{e_s \to e_{s\max}} \frac{|A_s|^2 \langle i|s\rangle \langle s|j\rangle}{|A_s|^2 \langle i'|s\rangle \langle s|j'\rangle} = C_{pp'}^S, \tag{39}$$

therefore

$$\lim_{L\to\infty} M_{pS}(L)/M_{p'S}(L) = C_{pp'}^S, \tag{40}$$

which is the result we desired.

In the case of a continuous spectrum our previous estimate for the approach to the limit breaks down because,

$$e_{\max-1}/e_{\max} = 1, \tag{41}$$

but, under the assumption that $|A_s|^2 \langle i|s\rangle \langle s|j\rangle$ is an analytic function of e_s in the vicinity of $e_{\max}$, we obtain,

$$\lim_{L\to\infty} M_{pS}(L)/M_{p'S}(L) = C_{pp'}^S + \text{const.}/L. \tag{42}$$

In the next section we give an example of this slow approach to the asymptote.

4. An example in one-dimension: maximal fluctuations

One of the most powerful applications of the theorem proved in the last section relates the moments of the transmission coefficient. We have shown in an earlier paper (Pendry & Barnes 1989) that the generalized transfer matrix for a one-dimensional system in the limit of $N = 0$ becomes

$$\boldsymbol{X}_{Sn}^{\otimes N=0} = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ r_n^- & (t_n^+)^2 & \cdot & \cdots \\ (r_n^-)^2 & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}. \tag{43}$$

Figure 5

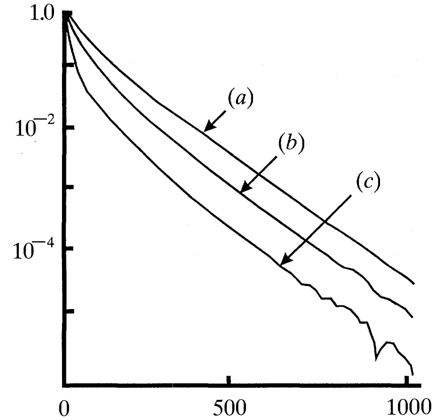


Figure 6

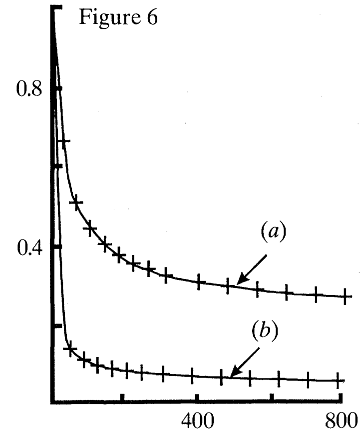


Figure 5. Moments of $|T_L|^2$ calculated by averaging over 9259648 one-dimensional samples of various lengths. The system had a bandwidth of 4, the energy for these calculations was $E = 1$, and the disorder of site energies was uniform with a width of 1. (*a*) $\langle |T_L|^2 \rangle$, (*b*) $\langle |T_L|^4 \rangle$, (*c*) $\langle |T_L|^{16} \rangle$.

Figure 6. Ratios of the moments of $|T_L|^2$ for a 1D system; see figure 5 for details of the simulations. The theoretical predictions are denoted by ' + ' symbols. Note the slow approach to the asymptotic value, approximately as $1/L$, indicating a continuous spectrum of eigenvalues. (*a*) $\langle |T_L|^4 \rangle / \langle |T_L|^2 \rangle$, (*b*) $\langle |T_L|^{16} \rangle / \langle |T_L|^2 \rangle$.

In the left-hand column of $\boldsymbol{X}_{Sn}^{\otimes N=0}$ we find all positive integer powers of r_n^-. By taking the cross product of $\boldsymbol{X}_{Sn}^{\otimes N=0}$ with its own complex conjugate we obtain a new transfer matrix, some elements of which are shown below,

$$\hat{\boldsymbol{X}}_n = (\boldsymbol{X}_{Sn}^{\otimes N=0}) \otimes (\boldsymbol{X}_{Sn}^{\otimes N=0})^* = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ |r_n^-|^2 & |t_n^+|^4 & \cdot & \cdots \\ |(r_n^-)|^4 & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{bmatrix}. \tag{44}$$

Hence we can express any power of the transmitted intensity as a linear combination of elements of $\hat{\boldsymbol{X}}_n$:

$$\left.\begin{aligned} |t_n^-|^2 &= 1 - |r_n^-|^2 = 1 - (\hat{\boldsymbol{X}}_n)_{2,1}, \\ |t_n^-|^4 &= 1 - 2\,|r_n^-|^2 + |r_n^-|^4 = 1 - 2(\hat{\boldsymbol{X}}_n)_{2,1} + (\hat{\boldsymbol{X}}_n)_{3,1}, \\ |t_n^-|^{2M} &= 1 - M\,|r_n^-|^2 + \cdots = 1 - M(\hat{\boldsymbol{X}}_n)_{2,1} + \cdots. \end{aligned}\right\} \tag{45}$$

We are now in a position to apply our theorem which states in this instance that

$$\lim_{L \to \infty} \langle |T_L^-|^{2M} \rangle / \langle |T_L^-|^2 \rangle = C(M), \tag{46}$$

where M is a constant independent of L.

This result is confirmed by a detailed theory (Kirkman & Pendry 1984*a*) which can give specific values to the $C(M)$'s, as was explained in §1. A numerical demonstration of this result was made by Pendry *et al.* (1990) for a disordered one-dimensional system and their data are displayed in figures 5 and 6. For this system the averaged transmitted intensity varies from $|T|^2 = 1$ for $L = 0$ to $|T|^2 \approx 5 \times 10^{-5}$ for $L = 1000$.

Table 1. *Comparison of values of $C(M) = \langle|T|^{2M}\rangle/\langle|T|^2\rangle$ as predicted by theory, and as calculated in a simulation*

M	$C(M)$ (theory)	$C(M)$ (simulation)
1	1.0	1.0
2	0.250	0.269
3	0.141	0.146
4	0.098	0.101
5	0.075	0.083
6	0.061	0.065

Despite this huge dynamic range the ratios of the moments tend to constants as we predict. The detailed analytic theory is also shown and is in excellent agreement with the simulations. The values of the constants obtained by extrapolating to $L = \infty$ are given in table 1.

We can apply our maximal fluctuation theorem to estimate the sample size needed to estimate an accurate average of $|T|^2$. According to this theorem the average will be dominated by a few samples which happen to have $|T|^2 \approx 1$. The average number of such samples in an ensemble of size N_{sam} is

$$N_{\text{sam}} \langle|T|^2\rangle, \tag{47}$$

therefore to obtain 5% accuracy in our estimate of $\langle|T|^2\rangle$ we require,

$$0.05 \approx 1/\surd(N_{\text{sam}}|T|^2). \tag{48}$$

In this instance the smallest values of $\langle|T|^2\rangle \approx 5\times10^{-5}$ so we require,

$$N_{\text{sam}} \approx 10^7. \tag{49}$$

This is the number of samples we have used and detailed inspection of figure 5 confirms that the fluctuations in our estimates remain good to the smallest values of $\langle|T|^2\rangle$ where they become just detectable at the 5% level.

However, the important message for this paper is that the ratios do indeed tend to constants.

Another point to be made is that $\hat{\boldsymbol{X}}_n$ is a transfer matrix with a continuous spectrum in the vicinity of $e_{\max}$, giving valuable confirmation of the more difficult of the two proofs we gave above. The continuous nature of the spectrum is evident in the smooth power law approach to the constant value.

5. Maximal fluctuations in two and three dimensions

For some time it has been apparent to use that the maximal fluctuations discussed in the previous section were a universal phenomenon, extending to two- and three-dimensional systems, and observable in localized or conducting systems. The method of universality classes provides a means of establishing these results on a firm theoretical basis. We shall present the theoretical proof first, then the numerical simulations.

The transmission matrix, $\boldsymbol{T}_L$, of a disordered system gives rise to a dimensionless conductance,

$$G_L = \text{trace}\ \boldsymbol{T}_L \boldsymbol{T}_L^{\dagger}. \tag{50}$$

The matrix $T_L T_L^{\dagger}$ has dimensions

$$D = L_x \tag{51}$$

in two dimensions, or

$$D = L_x \times L_y \tag{52}$$

in three dimensions, and its eigenvectors define a set of channels through which the current flows. The eigenvalues, λ, define the micro-conductance for each of these channels. They are constrained by unitarity to be real and,

$$0 \leqslant \lambda \leqslant 1. \tag{53}$$

It is with the statistics of these micro-conductances that we are concerned in this section: $P(\lambda)$, as defined by the moments,

$$\begin{aligned}\int_0^1 P(\lambda)\,\lambda^M\,\mathrm{d}\lambda &= D^{-1}\sum_{j=1}^{D}\langle\lambda_j^M\rangle\\ &= D^{-1}\,\mathrm{trace}\,\langle(T_L T_L^{\dagger})^M\rangle.\end{aligned} \tag{54}$$

Note the position of the brackets on the right-hand side of this equation.

We shall prove that these moments are all members of the same universality class and therefore their ratios tend to constants as $L\rightarrow\infty$.

We emphasize that maximal fluctuations in the micro-channels constitute a distinct phenomenon from the 'universal conductance fluctuations' discussed by Imry (1986), Al'tschuler (1985), and Lee & Stone (1985) which have to do with the fluctuations in the macroscopic (or sometimes mesoscopic) conductance, G_L, as defined by,

$$\langle(\mathrm{trace}\,T_L T_L^{\dagger})^2\rangle = \sum_{j=1}^{D}\sum_{j'=1}^{D}\langle\lambda_j\lambda_{j'}\rangle. \tag{55}$$

Again the positioning of the brackets is crucial. Universal conductance fluctuations are concerned with *correlations* between micro-channels; maximal fluctuations describe the statistics of a single micro-channel. Despite the name, universal conductance fluctuation theory is specific to delocalized systems which can be treated in perturbation theory. In one dimension, and in localized systems, the correlations between micro-channels change and a different sort of fluctuation is observed. In contrast the theorems on maximal fluctuations we prove here are obeyed in all dimensions in both localized and delocalized systems.

As in the one-dimensional case, T_L and R_L are related by (see, for example, Pendry 1990),

$$T_L T_L^{\dagger} + R_L R_L^{\dagger} = I, \tag{56}$$

except that in two and three dimensions this becomes a matrix equation. Thus any power of the matrix $T_L T_L^{\dagger}$ can be expressed as a sum of powers of $R_L R_L^{\dagger}$. It has been shown (Pendry 1990; Barnes & Pendry 1991) that there is a two- and three-dimensional equivalent of the transfer matrix containing $|r_n^-|^{2M}$ (equation (44)),

$$\begin{aligned}\hat{X}_n &= (X_{Sn}^{\otimes N=0})\otimes(X_{Sn}^{\otimes N=0})^*\\ &= \begin{bmatrix} 1 & 0 & 0 & \cdots\\ r^{-n}\otimes r_n^{-*} & t_n^+\otimes t_n^{+*}\otimes t_n^+\otimes t_n^{+*} & \cdot & \cdots\\ r_n^-\otimes r_n^{-*}\otimes r_n^-\otimes r_n^{-*} & \cdot & \cdot & \cdots\\ \cdot & \cdot & \cdot & \cdots\end{bmatrix}.\end{aligned} \tag{57}$$

In this expression $\boldsymbol{r}_n^-$ and $\boldsymbol{t}_n^+$ are matrices. Thus the quantities that interest us are contained in a long vector occupying the first column of $\hat{\boldsymbol{X}}_n$ and therefore belong to the same universality class. Hence the two- and three-dimensional form of our theorem states that,

$$\text{trace}\,\langle(\boldsymbol{T}_L\,\boldsymbol{T}_L^\dagger)^M\rangle/\text{trace}\,\langle\boldsymbol{T}_L\,\boldsymbol{T}_L^\dagger\rangle = C(M). \tag{58}$$

By implication, from (54),

$$\int_0^1 P(\lambda)\,\lambda^M\,\mathrm{d}\lambda\Big/\int_0^1 P(\lambda)\,\lambda\,\mathrm{d}\lambda = C(M), \tag{59}$$

and hence the conductances in the micro-channels exhibit the characteristic maximal fluctuations. In a two- or three-dimensional specimen of large cross section (large D), there will be very many micro-channels, of which a fraction,

$$N_{\text{cond}}/D \approx G_{L'}/D \tag{60}$$

will be highly conducting with transmission coefficient $\lambda \approx 1$, and the rest will be essentially in an insulating state.

It is worth relating these microscopic numbers to macroscopic variables. A cube, side 1 cm, of disordered material has typically as many micro-channels as there atoms in the cross section: roughly 10^{16} in this case and, typically, resistance between two faces of the cube might be 1 Ω. Our theorem states that conductance arises from N_{cond} micro-channels each with the maximum conductance of

$$e^2/h \approx 3.87\times10^{-5}\,\Omega^{-1}. \tag{61}$$

Thus out of the immense total number of micro-channels, 10^{16}, only 2.5×10^4 are actually responsible for the current!

6. Method of simulation

In the previous sections we have used the transfer matrix as a powerful analytical tool for investigation of scattering from disordered media. It is also in fact the basis of very successful numerical algorithms for the study of the metal–insulator transition and similar phenomena (MacKinnon & Kramer 1981, 1983). We now discuss an adaptation of this algorithm to the present problem and show some numerical results.

The calculations were based on the usual *Anderson* hamiltonian,

$$H = \sum_i \epsilon_i\,|i\rangle\langle i| + V\sum_{ij}|i\rangle\langle j|, \tag{62}$$

where is run over sites on a square or cubic lattice and the js are restricted to the nearest neighbours of i. In all the calculations the ϵ_is are independent random variables uniformly distributed in the range $-\frac{1}{2}W < \epsilon_i < \frac{1}{2}W$. For convenience the units are chosen such that the constant off-diagonal element V is unity.

The hamiltonian (62) may be rewritten in transfer matrix form by dividing the system into slices along one direction,

$$\begin{bmatrix}\boldsymbol{a}_{n+1}\\ \boldsymbol{a}_n\end{bmatrix} = \begin{bmatrix}\boldsymbol{E}-\boldsymbol{H}_n & -\boldsymbol{I}\\ \boldsymbol{I} & \boldsymbol{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{a}_n\\ \boldsymbol{a}_{n-1}\end{bmatrix} \equiv \boldsymbol{X}_n\begin{bmatrix}\boldsymbol{a}_n\\ \boldsymbol{a}_{n-1}\end{bmatrix}, \tag{63}$$

where $\boldsymbol{H}_n$ is the hamiltonian of the nth isolated slice and $\boldsymbol{a}_n$ is a vector representing the wave function on the sites of the nth slice.

To calculate the transmission matrix $\boldsymbol{t}_n$ from the transfer matrix $\boldsymbol{X}_n$ we first note that the left-handed eigenvectors of $\boldsymbol{X}_n$ are related to the right-handed ones by,

$$\boldsymbol{U}^{\text{left}} = \boldsymbol{U}^{\mathrm{T}}\boldsymbol{L}, \tag{64}$$

where $\boldsymbol{L}$ is a $2D \times 2D$ diagonal matrix with 1 in its top half and -1 in the bottom half. If we define $\boldsymbol{U}_0$ as the $2D \times D$ matrix whose columns form the complete set of right-handed eigenvectors corresponding to wave travelling to the right and normalized using (64) to unit current, then the transmission matrix corresponding to $\boldsymbol{X}_L$ may be defined as,

$$\boldsymbol{T}_L^{-1} = \boldsymbol{U}_0^{\mathrm{T}}\boldsymbol{L}\cdot\boldsymbol{X}_L\,\boldsymbol{U}_0. \tag{65}$$

This is, of course, identical to the definition in (6) above. We now have the basis of an algorithm for the calculation of $\boldsymbol{T}^{-1}$.

1. Choose a complete set of waves travelling to the right and generate $\boldsymbol{U}_0$.
2. Multiply this by the $\boldsymbol{X}_n$s to generate $\boldsymbol{Y}_L = \boldsymbol{X}_L\,\boldsymbol{U}_0$, which corresponds to the first column of (6).
3. Project out $\boldsymbol{T}_L^{-1}$ by multiplying by $\boldsymbol{U}_0^{\mathrm{T}}\boldsymbol{L}$.

Unfortunately, in practice, it is not possible to calculate $\boldsymbol{T}_L$ in this way. $\boldsymbol{T}_L^{-1}$ is so dominated by its largest eigenvalues that the smaller eigenvalues tend to be lost due to the finite accuracy of any numerical process. It is just these eigenvalues which are required in the calculation of $\boldsymbol{T}_L$ itself. To overcome this difficulty the algorithm must be modified.

Consider the quantity $\boldsymbol{Y}'_L$ defined by multiplying $\boldsymbol{Y}_L$ from the right by the inverse of the top half of $\boldsymbol{Y}_L$,

$$\boldsymbol{Y}'_L = \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{Y}'_{2L} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Y}_{1L} \\ \boldsymbol{Y}_{2L} \end{bmatrix}[\boldsymbol{Y}_{1L}]^{-1}, \tag{66}$$

where the subscripts 1 and 2 refer to the top and bottom halves of the $2\mathrm{D} \times \mathrm{D}$ matrix. The transmission matrix $\boldsymbol{T}_L$ can be recovered by solving the equation,

$$\boldsymbol{T}_L[\boldsymbol{U}_0^{\mathrm{T}}\boldsymbol{L}\boldsymbol{Y}'_L] = [\boldsymbol{Y}_{1L}]^{-1}. \tag{67}$$

Using (66) and (67) the unstable form of the algorithm can be modified as follows:

1. Every few iterations multiply $\boldsymbol{Y}$ from the right by the inverse of its top half. This is required every 10 to 14 iterations depending on the word length of the computer.
2. Store the product of the $\boldsymbol{Y}_1^{-1}$s separately.
3. Calculate $\boldsymbol{T}$ by solving (67).

Having calculated $\boldsymbol{T}$ it is a trivial matter to calculate trace $(\boldsymbol{T}\boldsymbol{T}^{\dagger})^M$.

7. Results of simulations

The simulations were performed for a range of parameters and system sizes in one, two and three dimensions. The theorem was particularly thoroughly tested in one dimension where the results in figures 5 were averaged over almost 10^7 realizations of the disordered system. The agreement between theory and the simulations is particularly impressive in this case.

In two and three dimensions we have concentrated on the (quasi) extended régime. Figure 7 shows trace $(\boldsymbol{T}\boldsymbol{T}^{\dagger})^M/\mathrm{trace}\,\boldsymbol{T}\boldsymbol{T}^{\dagger}$ for $W = 3.0$ and square samples ranging in

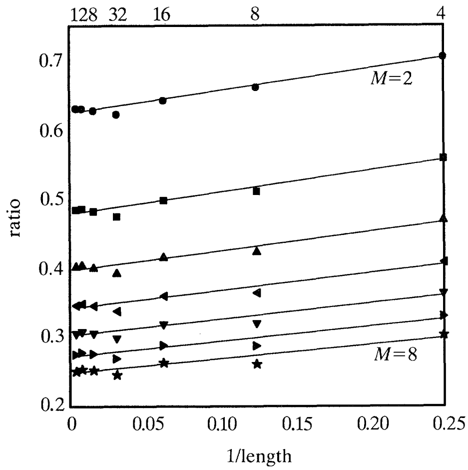


Figure 7. trace $(\boldsymbol{TT}^{\dagger})^M$/trace $\boldsymbol{TT}^{\dagger}$ against inverse length for squares of size $4 \leqslant L \leqslant 256$ and $E = 1.0$ and $W = 3.0$ averaged over 128 samples.

size from $L = 4$ to $L = 256$, and averaged over 128 samples. In this range the localization length (MacKinnon & Kramer 1983) is estimated to be of order 10^4. The data clearly tend towards finite values with increasing L. It is also particularly noticeable that the various lines for different M are parallel, implying an M-independent approach to the asymptotic value. It should be pointed out that, strictly speaking, the data are not in the asymptotic régime, in the sense that our systems are still orders of magnitude smaller than the localization length. The data presented here are characteristic of mesoscopic and weakly localized rather than strongly localized systems.

In three dimensions (figures 8 and 9) the data are plotted for cubes of size $4 \leqslant L \leqslant 20$ and disorder $W = 10$ and $W = 16.5$. The metal–insulator transition occurs at $W = 16.5$ (MacKinnon & Kramer 1983). Again there is a clear tendency towards finite asymptotic values and M-independent approach to these values. Notice that the gradient of the lines is now negative.

8. Universality classes defined by translational symmetry: an example in two dimensions

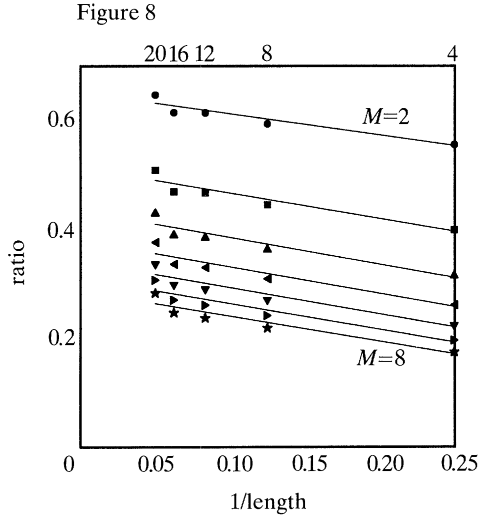
In many instances disordered samples retain some translational symmetry in the sense that the *distribution* of disorder is translationally invariant. We can use this symmetry to factorize the averaged transfer matrices and obtain new universality classes characterized by a wavevector $\boldsymbol{q}$. As an illustration of this residual of Bloch's theorem we consider the following quantities formed from the transmission matrix,

$$Tr(\boldsymbol{q},M) = \text{trace}\,[\boldsymbol{TT}_{\boldsymbol{q}}^{\dagger}(\boldsymbol{TT}^{\dagger})^M], \tag{68}$$

where the matrix $\boldsymbol{T}_{\boldsymbol{q}}^{\dagger}$ is defined by

$$(\boldsymbol{T}_{\boldsymbol{q}}^{\dagger})_{\boldsymbol{kk}'} = \boldsymbol{T}_{\boldsymbol{k}+\boldsymbol{q}\boldsymbol{k}'+\boldsymbol{q}}^{\dagger}. \tag{69}$$

Such quantities are a direct generalization of the trace$[(\boldsymbol{TT}^{\dagger})^M]$, and reduce to this form when $\boldsymbol{q}$ is set to zero. The $Tr(\boldsymbol{q},M)$, for $M = 0, 1, 2, \ldots$, can all be expressed in terms of elements of the same transfer matrix, and therefore translational invariance



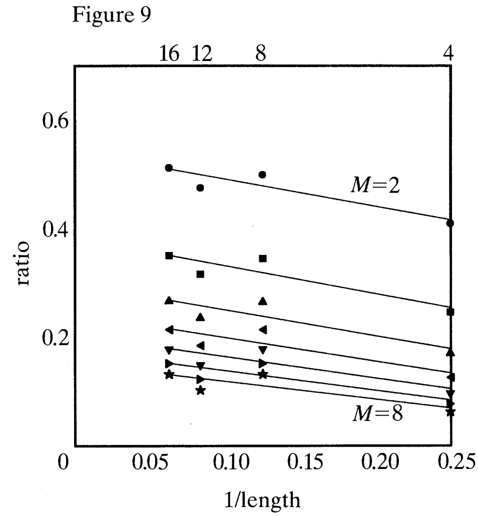


Figure 8. trace $(\boldsymbol{TT}^{\dagger})^{M}/\text{trace }\boldsymbol{TT}^{\dagger}$ against inverse length for cubes of size $4 \leqslant L \leqslant 20$ and $E = 1.0$ and $W = 10.0$ averaged over 128 samples.

Figure 9. trace $(\boldsymbol{TT}^{\dagger})^{M}/\text{trace }\boldsymbol{TT}^{\dagger}$ against inverse length of cubes of size $4 \leqslant L \leqslant 16$ and $E = 1.0$ and $W = 16.5$ averaged over 128 samples.

of the averaged system implies that the $Tr(\boldsymbol{q},M)$ separate into classes labelled by $\boldsymbol{q}$. It is thus expected that for a given $\boldsymbol{q}$ the ratios tend to constant values in the long-length limit,

$$\lim_{L\to\infty} \frac{\langle Tr(\boldsymbol{q},M)\rangle}{\langle Tr(\boldsymbol{q},0)\rangle} = C(M,\boldsymbol{q}). \tag{70}$$

The transmission matrix was calculated for a given member of the ensemble as a function of length L. From this, $\boldsymbol{TT}^{\dagger}$ and $\boldsymbol{TT}^{\dagger}_{q}$ are explicitly calculated. The eigenvectors and eigenvalues of the hermitian $\boldsymbol{TT}^{\dagger}$ are then found. Writing,

$$\boldsymbol{TT}^{\dagger} = \boldsymbol{S}^{\dagger}\boldsymbol{ES}, \tag{71}$$

the $Tr(\boldsymbol{q},M)$ for width $= 0, 1, \ldots, 7$ are calculated as,

$$Tr(\boldsymbol{q},M) = \text{trace}\,[\boldsymbol{STT}^{\dagger}_{q}\boldsymbol{S}^{\dagger}\,\boldsymbol{E}^{M}]. \tag{72}$$

An ensemble average is then performed. Because of the extra 'degree of freedom' associated with the imaginary part of the $Tr(\boldsymbol{q},M)$, these quantities fluctuate far more than the trace$[(\boldsymbol{TT}^{\dagger})^{M}]$. This means that the sample sizes for which converged averages can be calculated are limited to being small. A two-dimensional system of width 7 was chosen, and stable averages obtained up to a length of about 20. The other parameters were chosen to be:

energy, $E = 1.0$,

width of distribution, $W = 3.0$,

q-value, $\boldsymbol{q} = q = 1$: fluctuations increase with q.

An average over 500000 samples was performed. Figure 10 shows the real part of $Tr(\boldsymbol{q},M)$ against length for $M = 0.2$, and 7. Figure 11 displays the real part of the ratios for $M = 0, 1, \ldots, 7$ against length. Figure 12 shows the same data as a function

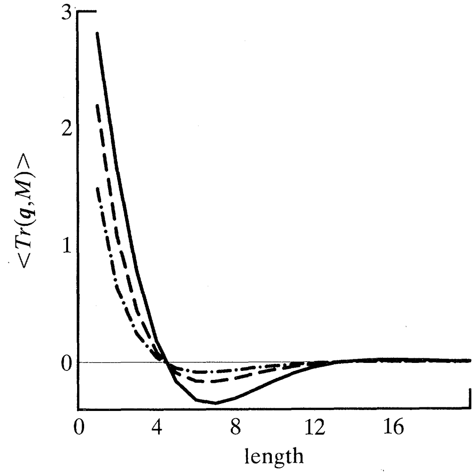


Figure 10. $Tr(q,M)$ as defined in the text. The average was taken over 500000 samples for a two-dimensional system of width 7, $E = 1$, $q = 1$, and the disorder of site energies was uniform with a width of 3. —·1 —, $M = 7$; ---, $M = 2$; ——, $M = 0$.

Figure 11

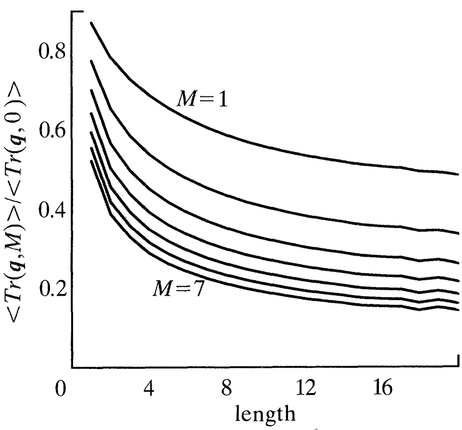


Figure 12

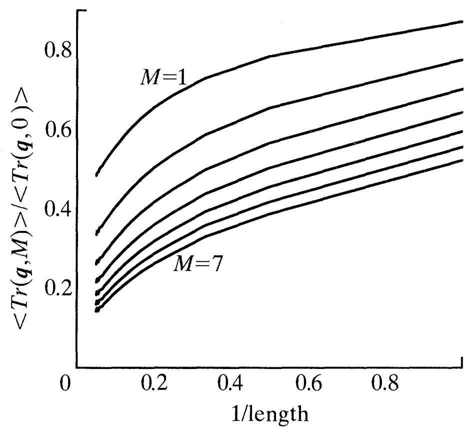


Figure 11. For a fixed $\boldsymbol{q}$, all the $Tr(\boldsymbol{q},M)$ should belong to the same universality class, according to our theory, which implies that their ratios tend to a constant with increasing length of system, L. Figure 12. The same as for figure 11, except that the coordinate is now $1/L$. The prediction is that plots should be linear in this variable near the origin.

of $1/L$. The imaginary parts were found to be at least two orders of magnitude smaller. The oscillatory nature of the $Tr(\boldsymbol{q},M)$ as a function of length is of course due to the non-zero $\boldsymbol{q}$: the oscillation frequency increases with $|\boldsymbol{q}|$ (for $|\boldsymbol{q}| < \pi$) and is energy dependent. The evolution of the $Tr(\boldsymbol{q},M)$ might be described by the form,

$$\int \exp\,(-\alpha_k L)\,\exp\,[\mathrm{i}K_k - K_{k+q})L]\,\mathrm{d}\alpha_k, \tag{73}$$

which is consistent with the above observations.

Examining figure 12 we see that the ratios are all clearly heading towards limiting values with increasing length, confirming the prediction that the $Tr(\boldsymbol{q},M)$ all belong to the same universality class.

9. Conclusions

Waves transmitted through disordered media present major challenges to theorists trying to describe their statistics. We have introduced a new concept into the debate: that of *universality classes*. Each member of a universality class has the same asymptotic dependence on length of the medium and the classification can be made on very general considerations without knowing details of the medium's structure. Hence the use of the word *universality*. We hope that this concept will be of value in proving very general theorems about fluctuations, and in this paper we have applied it to proving the existence of *maximal fluctuations* in the transmission coefficients through disordered media: basically the transmission channels are either completely transparent, or completely cut off. This surprising result was confirmed by a variety of simulations in one, two and three dimensions.

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