Cylindrical current sources

Ben Wood

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1 The question of 'external' currents

Our problem is to determine the field radiated by a given current source. Maxwell's equations give us

$$\nabla^2 \mathbf{A} - \nabla(\nabla \cdot \mathbf{A}) - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \epsilon\mu \nabla \frac{\partial \phi}{\partial t} \tag{1}$$

$$\nabla^2 \phi + \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} = -\frac{\rho}{\epsilon} \tag{2}$$

with

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{3}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}. \tag{4}$$

We are then able to select a suitable gauge. The choice suggested by Panofsky and Phillips [2] is

$$\nabla \cdot \mathbf{A} + \epsilon\mu \frac{\partial \phi}{\partial t} + \mu\sigma\phi = 0. \tag{5}$$

This is the Lorentz condition for conducting media. Applying it to equation 2 gives

$$\nabla^2 \phi - \epsilon\mu \frac{\partial^2 \phi}{\partial t^2} - \mu\sigma \frac{\partial \phi}{\partial t} = -\frac{\rho}{\epsilon}. \tag{6}$$

Slightly more work is required when applying the gauge condition to equation 1. We begin by rearranging the terms slightly:

$$\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla \left(\nabla \cdot \mathbf{A} + \epsilon\mu \frac{\partial \phi}{\partial t} \right) \tag{7}$$

The gauge condition, equation 5, may then be used to replace the final term:

$$\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} - \mu\sigma \nabla\phi. \tag{8}$$

Finally, we add a term to both sides; the left-hand side is then the vector equivalent of equation 6:

$$\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu\sigma \frac{\partial \mathbf{A}}{\partial t} = -\mu \mathbf{J} - \mu\sigma \left(\nabla\phi + \frac{\partial \mathbf{A}}{\partial t} \right). \tag{9}$$

We can identify the term in brackets as $-\mathbf{E}$, giving the result

$$\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} - \mu\sigma \frac{\partial \mathbf{A}}{\partial t} = -\mu \left(\mathbf{J} - \sigma \mathbf{E} \right). \tag{10}$$

The source for the vector potential is now the difference between $\mathbf{J}$ and $\sigma\mathbf{E}$. With the usual definitions for the conductivity, current and field, this would be zero.

The argument of Panofsky and Phillips is that as long as we confine our attention to the fields arising from the current, rather than the *external* fields which produce the current, we can write

$$\mathbf{J}' = \mathbf{J} - \sigma \mathbf{E} \tag{11}$$

where $\mathbf{J}'$ is the external current and $\mathbf{E}'$ is the external field, with

$$\mathbf{J}' = \sigma \mathbf{E}'. \tag{12}$$

However, there appears to be a problem with this line of reasoning. Consider substituting equation 12 into equation 11; this gives

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}'), \tag{13}$$

implying that $\mathbf{E}$, ϕ and $\mathbf{A}$ do not represent the total fields, since they do not include the external components. Why, then, would we use the total current $\mathbf{J}$ in equation 1? This seems to be inconsistent.

It is possible to retain equation 11, as long as equation 12 is discarded and replaced with the alternative

$$\mathbf{J}' = \sigma' \mathbf{E}. \tag{14}$$

The separation has been transferred to the conductivity, some of which (σ') is now considered to be 'external'; the total field $\mathbf{E}$ is retained. Ohm's Law, when applied to the total fields and currents, now takes the form

$$\mathbf{J} = (\sigma + \sigma')\mathbf{E}. \tag{15}$$

It is then evident that the conductivity which appears in the decoupled vector and scalar potential equations is not the full conductivity; specifically, the part relating to the external current has been removed.

In the absence of additional currents ($\sigma = 0$), the relations become

$$\nabla^2 \phi - \epsilon\mu \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{16}$$

$$\nabla^2 \mathbf{A} - \epsilon\mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}'. \tag{17}$$

In free space we can also replace $\epsilon\mu$ with $1/c^2$.

2 Waves emitted by a cylindrical source

We will consider two kinds of source here. In both cases, the current is confined to a thin shell of radius r_0.

2.1 Axial current

In our first system, we consider the following current source:

$$\mathbf{J} = \frac{I_0 \hat{\mathbf{z}}}{2\pi r_0} \delta(r - r_0) e^{ik_z z - i\omega t}. \tag{18}$$

The current flows in the z-direction only; $\mathbf{A}$ is therefore also parallel to z. We require a solution to the equation

$$\nabla^2 A_z - \frac{1}{c^2} \frac{\partial^2 A_z}{\partial t^2} = -\frac{I_0 \mu_0}{2\pi r_0} \delta(r - r_0) e^{ik_z z - i\omega t}. \tag{19}$$

We can immediately guess the form of the z- and t-dependence of A_z. The function

$$A_z = f(r, \theta) e^{ik_z z - i\omega t} \tag{20}$$

satisfies equation 19 as long as

$$\left(\nabla_\parallel^2 + \frac{\omega^2}{c^2} - k_z^2 \right) f = -\frac{I_0 \mu_0}{2\pi r_0} \delta(r - r_0). \tag{21}$$

The homogeneous version of this equation is

$$\left(\nabla_\parallel^2 + \frac{\omega^2}{c^2} - k_z^2 \right) f = 0, \tag{22}$$

which we can solve by separation of variables:

$$f(r, \theta) = R(r)\Theta(\theta). \tag{23}$$

This gives

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial (R\Theta)}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 (R\Theta)}{\partial \theta^2} + \kappa^2 R\Theta = 0 \tag{24}$$

where

$$\kappa^2 = \frac{\omega^2}{c^2} - k_z^2. \tag{25}$$

In these notes, we assume that $k_z c < \omega$, so that κ is always real. Dividing equation 24 by f and rearranging leads to the separated form

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{dR}{dr}\right) + \kappa^2 r^2 = -\frac{1}{\Theta}\frac{d\Theta}{d\theta}. \tag{26}$$

We therefore set both sides of the equation to be equal to m^2, the separation constant. By inspection, the solution for Θ is

$$\Theta = e^{\pm im\theta}. \tag{27}$$

In order for the function to be single-valued, we require that m be an integer. The R equation is

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + (\kappa^2 r^2 - m^2)R = 0, \tag{28}$$

which is a version of Bessel's equation, and has the solution

$$R = R_1 H_m^{(1)}(\kappa r) + R_2 H_m^{(2)}(\kappa r). \tag{29}$$

Our inhomogeneous equation does not depend on θ, and we therefore require that $m = 0$. The full solution for f is then

$$f(r) = R_1 H_0^{(1)}(\kappa r) + R_2 H_0^{(2)}(\kappa r). \tag{30}$$

We expect the constants R_1 and R_2 to be different in the two regions $r < r_0$ and $r > r_0$ (separated by the current sheet). The field must have a gradient discontinuity at the boundary; this can be seen by integrating equation 21 about $r = r_0$:

$$\begin{aligned} \int_{r_0-\Delta}^{r_0+\Delta} 2\pi r\left(\nabla_\parallel^2 + \kappa^2\right) f(r)\, dr &= -\frac{I_0\mu_0}{2\pi r_0}\int_{r_0-\Delta}^{r_0+\Delta} 2\pi r\delta(r - r_0)\, dr \\ \left[2\pi r\frac{df}{dr}\right]_{r_0-\Delta}^{r_0+\Delta} + 2\pi\kappa^2\int_{r_0-\Delta}^{r_0+\Delta} r f(r)\, dr &= -I_0\mu_0. \end{aligned} \tag{31}$$

Letting $\Delta \to 0$ then gives

$$\left.\frac{df}{dr}\right|_{r=r_0^+} - \left.\frac{df}{dr}\right|_{r=r_0^-} = -\frac{I_0\mu_0}{2\pi r_0}. \tag{32}$$

We will use this boundary condition later.

The asymptotic form of the zeroth-order Hankel functions as the argument tends to ∞ is

$$H_0^{(1)}(\kappa r) \sim (1-i)\sqrt{\frac{1}{\pi\kappa r}}e^{i\kappa r} \tag{33}$$

$$H_0^{(2)}(\kappa r) \sim (1+i)\sqrt{\frac{1}{\pi\kappa r}}e^{-i\kappa r}. \tag{34}$$

An outgoing wave is represented by $H_0^{(1)}$, while $H_0^{(2)}$ represents an incoming wave; the solution for $r > r_0$ must therefore contain only $H_0^{(1)}$.

When $r = 0$, the field must not diverge. Both $H_0^{(1)}$ and $H_0^{(2)}$ diverge logarithmically at the origin; as $r \to 0$, we have

$$H_0^{(1)}(\kappa r) \sim \frac{2i}{\pi}\ln(\kappa r) \tag{35}$$

$$H_0^{(2)}(\kappa r) \sim -\frac{2i}{\pi}\ln(\kappa r). \tag{36}$$

The non-divergent combination of the two functions is

$$H_0^{(1)}(\kappa r) + H_0^{(2)}(\kappa r) = 2J_0(\kappa r), \tag{37}$$

the Bessel function of the first kind. Our boundary conditions at the origin and at infinity have shown that

$$f(r) = A_0 \begin{cases} \alpha J_0(\kappa r) & r < r_0 \\ H_0^{(1)}(\kappa r) & r > r_0 \end{cases} \tag{38}$$

where α and A_0 are constants to be determined. Our next boundary condition is that the field must be continuous at $r = r_0$, so that

$$\alpha = \frac{H_0^{(1)}(\kappa r_0)}{J_0(\kappa r_0)} = 1 + i\frac{Y_0(\kappa r_0)}{J_0(\kappa r_0)}. \tag{39}$$

Finally, we may use the gradient discontinuity on the boundary (equation 32) to give

$$\begin{aligned}
-\frac{I_0\mu_0}{2\pi r_0} &= A_0\left(-\kappa H_1^{(1)}(\kappa r_0) + \alpha\kappa J_1(\kappa r_0)\right) \\
&= A_0\kappa\left(-J_1(\kappa r_0) - iY_1(\kappa r_0) + \frac{J_0(\kappa r_0) + iY_0(\kappa r_0)}{J_0(\kappa r_0)}J_1(\kappa r_0)\right) \\
&= iA_0\kappa\left(-Y_1(\kappa r_0) + \frac{Y_0(\kappa r_0)}{J_0(\kappa r_0)}J_1(\kappa r_0)\right) \\
A_0 &= \frac{iI_0\mu_0}{2\pi\kappa r_0}\left(-Y_1(\kappa r_0) + \frac{Y_0(\kappa r_0)}{J_0(\kappa r_0)}J_1(\kappa r_0)\right)^{-1}.
\end{aligned} \tag{40}$$

This relates the amplitude of the outgoing wave to the current which produces it. The full solution is

$$A_z = A_0 e^{ik_z z - i\omega t} \begin{cases} \alpha J_0(\kappa r) & r < r_0 \\ H_0^{(1)}(\kappa r) & r > r_0. \end{cases} \tag{41}$$

In the limit of small r_0,

$$\begin{aligned} J_0(\kappa r_0) &\sim 1 &(42)\\ J_1(\kappa r_0) &\sim \frac{\kappa r_0}{2} &(43)\\ Y_0(\kappa r_0) &\sim \frac{2}{\pi}\ln(\kappa r_0) &(44)\\ Y_1(\kappa r_0) &\sim -\frac{2}{\pi \kappa r_0}. &(45) \end{aligned}$$

Considering only the dominant terms, we then have

$$\begin{aligned} A_0 &\approx \frac{iI_0\mu_0}{4} \\ \alpha &\approx \frac{2i}{\pi}\ln(\kappa r_0), &(46) \end{aligned}$$

and the full solution becomes

$$A_z = \frac{iI_0\mu_0}{4} e^{ik_z z - i\omega t} \begin{cases} \frac{2i}{\pi}\ln(\kappa r_0) J_0(\kappa r) & r < r_0 \\ H_0^{(1)}(\kappa r) & r > r_0. \end{cases} \tag{47}$$

Next, we calculate the magnetic and electric fields. The magnetic field is

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ &= -\hat{\boldsymbol{\phi}} \frac{\partial A_z}{\partial r} \\ &= \hat{\boldsymbol{\phi}} A_0 \kappa e^{ik_z z - i\omega t} \begin{cases} \alpha J_1(\kappa r) & r < r_0 \\ H_1^{(1)}(\kappa r) & r > r_0. \end{cases} &(48) \end{aligned}$$

The electric field can then be calculated either from equations 4 and 5 or directly from Maxwell's equations. Taking the latter approach, we start with

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{49}$$

The time dependence is harmonic and there is no current away from the source, so we can rearrange this equation to give

$$\mathbf{E} = -\frac{c^2}{i\omega}\nabla \times \mathbf{B}. \tag{50}$$

The form of $\mathbf{B}$ means that this expression reduces to

$$\mathbf{E} = -\frac{c^2}{i\omega}\left[-\hat{\mathbf{r}}\frac{\partial B_\phi}{\partial z} + \hat{\mathbf{z}}\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi)\right]. \tag{51}$$

Substituting for $\mathbf{B}$, we have for the inner region

$$\begin{aligned}\mathbf{E}(r < r_0) &= -\frac{c^2\alpha A_0\kappa}{i\omega}\left[-\hat{\mathbf{r}}ik_zJ_1(\kappa r)\right.\\ &\quad \left.+\hat{\mathbf{z}}\left(\frac{J_1(\kappa r)}{r} + \kappa\left[J_0(\kappa r) - \frac{1}{\kappa r}J_1(\kappa r)\right]\right)\right]e^{ik_zz-i\omega t}\\ &= \alpha A_0\kappa c\left[\hat{\mathbf{r}}\left(\frac{k_zc}{\omega}\right)J_1(\kappa r) + \hat{\mathbf{z}}\left(\frac{i\kappa c}{\omega}\right)J_0(\kappa r)\right]e^{ik_zz-i\omega t}. \end{aligned}\tag{52}$$

Similarly, the result for the outer region is

$$\mathbf{E}(r > r_0) = A_0\kappa c\left[\hat{\mathbf{r}}\left(\frac{k_zc}{\omega}\right)H_1^{(1)}(\kappa r) + \hat{\mathbf{z}}\left(\frac{i\kappa c}{\omega}\right)H_0^{(1)}(\kappa r)\right]e^{ik_zz-i\omega t}. \tag{53}$$

2.2 Solenoidal current

We now move on to the second system, in which the current flows around the z-axis:

$$\mathbf{J} = J_0\hat{\boldsymbol{\phi}}\delta(r - r_0)e^{ik_zz-i\omega t}. \tag{54}$$

Once again, $\mathbf{A}$ must satisfy the vector potential equation (equation 17). Based on the results of the previous section, we can guess a solution of the following form:

$$\begin{aligned}\mathbf{A} &= \hat{\boldsymbol{\phi}}A_\phi(r,z)\\ &= A_0\hat{\boldsymbol{\phi}}e^{ik_zz-i\omega t}\begin{cases}\alpha J_n(\kappa r_0) & r < r_0\\ H_n^{(1)}(\kappa r_0) & r > r_0.\end{cases}\end{aligned}\tag{55}$$

Away from the current source at $r = r_0$, the vector potential equation reduces to the homogeneous form

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0 \tag{56}$$

As before, we will proceed by obtaining the solutions to this equation, and then applying appropriate boundary conditions; the current source will provide one of these conditions.

We first calculate the effect of the Laplacian operator on a vector of the type described in equation 55:

$$\nabla^2\left(\hat{\boldsymbol{\phi}}A_\phi(r,z)\right) = \hat{\boldsymbol{\phi}}\left(\nabla^2 A_\phi - \frac{A_\phi}{r^2}\right)$$

$$
\begin{aligned}
&= \hat{\boldsymbol{\phi}}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\phi}{\partial r}\right) + \frac{\partial^2 A_\phi}{\partial z^2} - \frac{A_\phi}{r^2}\right] \\
&= \hat{\boldsymbol{\phi}}\left[\frac{\partial^2 A_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial A_\phi}{\partial r} + \frac{\partial^2 A_\phi}{\partial z^2} - \frac{A_\phi}{r^2}\right]. \qquad (57)
\end{aligned}
$$

Next, we susbtitute for A_ϕ from equation 55, and use the recursion relations for differentiation of Bessel functions to obtain

$$
\nabla^2 \mathbf{A} = A_0 \hat{\boldsymbol{\phi}} e^{ik_z z - i\omega t}\left(-\kappa^2 - k_z^2 + \frac{n^2-1}{r^2}\right)\begin{cases} \alpha J_n(\kappa r_0) & r < r_0 \\ H_n^{(1)}(\kappa r_0) & r > r_0. \end{cases} \qquad (58)
$$

In order for $\mathbf{A}$ to satisfy equation 56, we therefore require that

$$
-\kappa^2 - k_z^2 + \frac{n^2-1}{r^2} + \frac{\omega^2}{c^2} = 0. \qquad (59)
$$

From the definition of κ (equation 25), we can see that the condition is met, as long as we set $n = 1$. (Note that $n = -1$ does not give an independent solution.)

The constant α is obtained from the requirement that $\mathbf{A}$ be continuous at $r = r_0$; the result is

$$
\alpha = \frac{H_1^{(1)}(\kappa r_0)}{J_1(\kappa r_0)}. \qquad (60)
$$

The final constant in our solution, A_0, is related to J_0; we can determine the relationship by using the same technique as before, ie. integrating equation 17 over a region around r_0. Only the ϕ-component of the equation is non-zero:

$$
\begin{aligned}
\int_{r_0-\Delta}^{r_0+\Delta} 2\pi r\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A_\phi}{\partial r}\right) + \left(\kappa^2 - \frac{1}{r^2}\right)A_\phi\right] dr &= -\mu_0 \int_{r_0-\Delta}^{r_0+\Delta} 2\pi r J_\phi \, dr \\
\left[r\frac{\partial A_\phi}{\partial r}\right]_{r_0-\Delta}^{r_0+\Delta} + \int_{r_0-\Delta}^{r_0+\Delta}\left(\kappa^2 r - \frac{1}{r}\right)A_\phi \, dr &= -\mu_0 r_0 J_0 e^{ik_z z - i\omega t}. \qquad (61)
\end{aligned}
$$

Since A_ϕ is continuous at $r = r_0$, the second term on the left-hand side disappears in the limit $\Delta \to 0$, leaving

$$
\left.\frac{\partial A_\phi}{\partial r}\right|_{r=r_0^+} - \left.\frac{\partial A_\phi}{\partial r}\right|_{r=r_0^-} = -\mu_0 J_0 e^{ik_z z - i\omega t}. \qquad (62)
$$

Substituting for A_ϕ then gives

$$
A_0 \kappa\left[H_0^{(1)}(\kappa r_0) - \frac{1}{\kappa r_0}H_1^{(1)}(\kappa r_0) - \alpha J_0(\kappa r_0) + \frac{\alpha}{\kappa r_0}J_1(\kappa r_0)\right] = -\mu_0 J_0 \qquad (63)
$$

$$
A_0 = -\frac{\mu_0 J_0}{i\kappa}\left[Y_0^{(1)}(\kappa r_0) - \frac{Y_1(\kappa r_0) J_0(\kappa r_0)}{J_1(\kappa r_0)}\right]^{-1}. \qquad (64)
$$

In the limit $r_0 \ll 1$, this becomes

$$A_0 \approx \frac{i\mu_0 \pi \kappa r_0^2 J_0}{4}. \tag{65}$$

Having obtained $\mathbf{A}$, we can now deduce the electric and magnetic fields. For convenience, only the fields in the outer region will be calculated; those in the inner region can then be obtained by replacing $H_n^{(1)}$ with J_n and multiplying the resultant field by α. We begin with the magnetic field:

$$\begin{aligned}
\mathbf{B}(r > r_0) &= \nabla \times \mathbf{A} \\
&= -\hat{\mathbf{r}}\frac{\partial A_\phi}{\partial z} + \hat{\mathbf{z}}\frac{1}{r}\frac{\partial}{\partial r}\left(rA_\phi\right) \\
&= A_0 e^{ik_z z - i\omega t}\Bigg(-\hat{\mathbf{r}} i k_z H_1^{(1)}(\kappa r) \\
&\quad + \hat{\mathbf{z}}\left[\frac{H_1^{(1)}(\kappa r)}{r} + \kappa H_0^{(1)}(\kappa r) - \frac{H_1^{(1)}(\kappa r)}{r}\right]\Bigg) \\
&= A_0 e^{ik_z z - i\omega t}\left(-\hat{\mathbf{r}} i k_z H_1^{(1)}(\kappa r) + \hat{\mathbf{z}}\kappa H_0^{(1)}(\kappa r)\right).
\end{aligned} \tag{66}$$

The electric field then follows:

$$\begin{aligned}
\mathbf{E}(r > r_0) &= -\frac{c^2}{i\omega}\nabla \times \mathbf{B} \\
&= -\frac{c^2}{i\omega}\hat{\boldsymbol{\phi}}\left(\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}\right) \\
&= -\frac{A_0 c^2}{i\omega}\hat{\boldsymbol{\phi}} e^{ik_z z - i\omega t}\left(k_z^2 H_1^{(1)}(\kappa r) + \kappa^2 H_1^{(1)}(\kappa r)\right) \\
&= i\omega A_0 \hat{\boldsymbol{\phi}} e^{ik_z z - i\omega t} H_1^{(1)}(\kappa r).
\end{aligned} \tag{67}$$

For this solenoidal current, the electric field is purely in the ϕ-direction, while the magnetic field has only r- and z-components. These fields are complementary to those obtained in the previous section; the two sets of fields are related by the transformation

$$\begin{aligned}
\mathbf{E} &\longleftrightarrow \mathbf{H} \\
\epsilon_0 &\longleftrightarrow -\mu_0.
\end{aligned} \tag{68}$$

In free space, Maxwell's equations are invariant under this transformation.

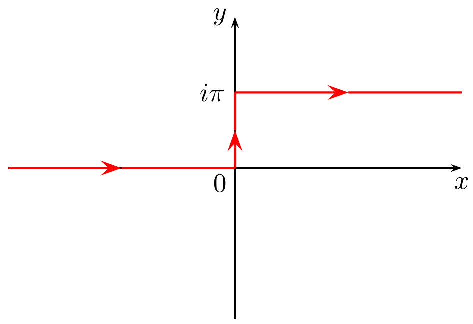

Figure 1: The contour of integration implied in equation 69.

3 Plane wave expansion of Hankel functions

Our aim is to expand the function $H_n^{(1)}(\kappa r)$ (where $n = 0, 1, 2, \ldots$) in terms of plane waves. We begin with the definition [1]

$$H_n^{(1)}(z) = \frac{1}{i\pi} \int_{-\infty}^{\infty + i\pi} e^{z \sinh t - nt} \, dt, \tag{69}$$

which is valid for $|\arg z| < \pi/2$, although we will be considering only real, strictly positive arguments. The integration contour is shown in figure 1. We then change variables to α, where

$$t = i\alpha + \frac{i\pi}{2}. \tag{70}$$

The integral becomes

$$\begin{aligned} H_n^{(1)}(z) &= \frac{1}{\pi} \int_{-\pi/2 + i\infty}^{\pi/2 - i\infty} \exp\left[\frac{z}{2} \left(e^{i\alpha + i\pi/2} - e^{-i\alpha - i\pi/2} \right) - n(i\alpha + i\pi/2) \right] d\alpha \\ &= \frac{e^{-in\pi/2}}{\pi} \int_{-\pi/2 + i\infty}^{\pi/2 - i\infty} \exp\left(iz \cos\alpha - in\alpha \right) d\alpha. \end{aligned} \tag{71}$$

The new contour is plotted in figure 2. In fact, we can deform this contour slightly without changing the value of the integral. If we move the entire contour to the left in the complex plane (as indicated in figure 2), we do not cross any singularities. For this procedure to be valid, we must also demonstrate that the value of the integrand at the endpoints is unchanged.

We first consider the upper limit; here, the value of the integrand is now

$$\lim_{\xi \to \infty} \left[\exp\left(iz \cos\alpha - in\alpha \right) \right]_{\alpha = \pi/2 - i\xi + \pi/2 - \phi}$$

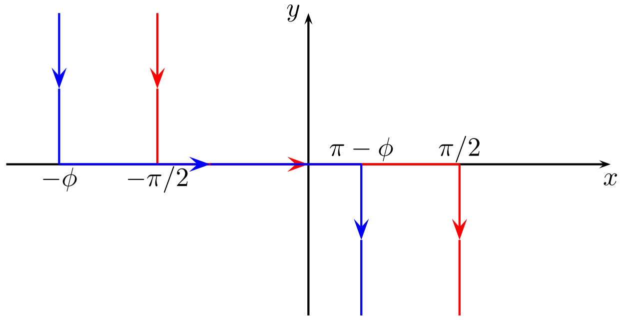

Figure 2: The new contour of integration used in equation 71 is plotted in red; the blue line shows the effect of displacing this contour by an amount $(\pi/2-\phi)$, where ϕ is a real number.

$$
\begin{aligned}
&= \lim_{\xi\to\infty} \exp\left[iz\cos\left(\pi - i\xi - \phi\right) - in\left(\pi - i\xi - \phi\right)\right] \\
&= \lim_{\xi\to\infty} \exp\left[\frac{iz}{2}e^{i(\pi-\phi)+\xi} - n\xi - in\left(\pi - \phi\right)\right] \\
&= \lim_{\xi\to\infty} \exp\left[-\frac{iz}{2}(\cos\phi - i\sin\phi)e^{\xi} - n\xi - in\left(\pi - \phi\right)\right] \\
&= e^{-in(\pi-\phi)} \lim_{\xi\to\infty} \exp\left(-\frac{z}{2}e^{\xi}\sin\phi - n\xi - \frac{iz}{2}e^{\xi}\cos\phi\right). \qquad (72)
\end{aligned}
$$

The behaviour is dominated by the first of the three ξ-dependent terms. When $\sin\phi > 0$,

$$\lim_{\xi\to\infty} \exp\left(-\frac{z}{2}e^{\xi}\sin\phi - n\xi - \frac{iz}{2}e^{\xi}\cos\phi\right) = 0, \qquad (73)$$

and the integrand is zero at the upper limit. This is the case when the contour is undeformed ($\phi = \pi/2$). If we restrict ϕ so that

$$0 < \phi < \pi \qquad (74)$$

then the value of the integrand at the upper limit remains zero; therefore we are free to move the upper limit by an amount $(\pi/2 - \phi)$, where ϕ lies within the range specified in equation 74.

The same analysis can be applied to the lower limit. The value of the integrand here is

$$
\begin{aligned}
&\lim_{\xi\to\infty} \left[\exp\left(iz\cos\alpha - in\alpha\right)\right]_{\alpha=-\pi/2+i\xi+\pi/2-\phi} \\
&\quad = \lim_{\xi\to\infty} \exp\left[iz\cos\left(i\xi - \phi\right) - in\left(i\xi - \phi\right)\right] \\
&\quad = \lim_{\xi\to\infty} \exp\left[\frac{iz}{2}e^{i\phi+\xi} + n\xi + in\phi\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\xi\to\infty} \exp\left[\frac{iz}{2}(\cos\phi + i\sin\phi)e^{\xi} + n\xi + in\phi\right] \\
&= e^{in\phi} \lim_{\xi\to\infty} \exp\left(-\frac{z}{2}e^{\xi}\sin\phi + n\xi + \frac{iz}{2}e^{\xi}\cos\phi\right). \qquad (75)
\end{aligned}
$$

As with the upper limit, the dominant term is

$$
-\frac{z}{2}e^{\xi}\sin\phi, \qquad (76)
$$

and the integrand evaluates to zero as long as ϕ lies in the range given in equation 74.

We have demonstrated that for this range of values of ϕ, the deformation of the contour does not alter the value of the integral.

We may therefore write

$$
H_n^{(1)}(z) = \frac{e^{-in\pi/2}}{\pi}\int_{i\infty-\phi}^{\pi-i\infty-\phi} \exp\left(iz\cos\alpha - in\alpha\right)\, d\alpha \qquad (77)
$$

and make the change of variables

$$
\theta = \alpha + \phi, \qquad (78)
$$

giving

$$
\begin{aligned}
H_n^{(1)}(z) &= \frac{e^{-in\pi/2}}{\pi}\int_{i\infty}^{\pi-i\infty} \exp\left(iz\cos(\theta-\phi) - in(\theta-\phi)\right)\, d\theta \qquad (79)\\
&= \frac{e^{-in\pi/2}}{\pi}\int_{i\infty}^{\pi-i\infty} \exp\left(iz\cos\theta\cos\phi + iz\sin\theta\sin\phi - in(\theta-\phi)\right)\, d\theta.
\end{aligned}
$$

In our original function, the argument of the Hankel function was κr. Replacing z with κr, and writing

$$
\begin{aligned}
x &= r\cos\phi \qquad (80)\\
y &= r\sin\phi \qquad (81)
\end{aligned}
$$

leads to the following expression:

$$
H_n^{(1)}(\kappa r) = \frac{e^{-in\pi/2}}{\pi}\int_{i\infty}^{\pi-i\infty} e^{i\kappa(x\cos\theta + y\sin\theta)}e^{-in(\theta-\phi)}\, d\theta. \qquad (82)
$$

Note that the restriction on the allowed values of ϕ has become a restriction on y:

$$
y > 0. \qquad (83)
$$

Finally, we change variables one more time. Our new integration variable is

$$
k_x = \kappa\cos\theta; \qquad (84)
$$

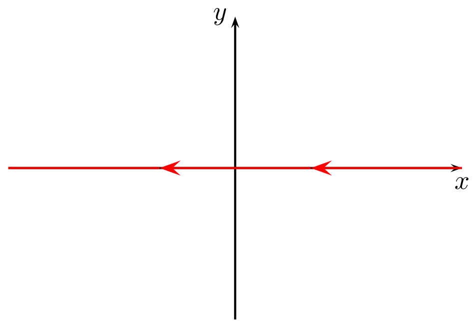

Figure 3: The contour of integration for the integral in equation 92.

the new contour is shown in figure 3. For convenience of notation, we also introduce a second variable:

$$k_y = \kappa \sin\theta \tag{85}$$

so that

$$\begin{aligned} dk_x &= -\kappa \sin\theta\, d\theta \\ &= -k_y\, d\theta. \end{aligned} \tag{86}$$

By following the value of $\sin\theta$ around the contour, and comparing k_x and k_y, we obtain the following choices for the sign of k_y:

$$k_y = \begin{cases} \sqrt{\kappa^2 - k_x^2} & \text{when } |k_x| \leq \kappa \\ i\sqrt{k_x^2 - \kappa^2} & \text{otherwise.} \end{cases} \tag{87}$$

Using the new variables, we also have

$$\kappa r \cos(\theta - \phi) = k_x x + k_y y \tag{88}$$

$$\kappa r \sin(\theta - \phi) = k_y x - k_x y, \tag{89}$$

giving

$$\begin{aligned} e^{-in(\theta-\phi)} &= [\cos(\theta-\phi) - i\sin(\theta-\phi)]^n \\ &= \left[\frac{k_x x + k_y y - i(k_y x - k_x y)}{\kappa r}\right]^n \\ &= \left[\frac{(k_x - ik_y)(x+iy)}{\kappa r}\right]^n. \end{aligned} \tag{90}$$

It may be more helpful to rewrite only the θ-dependent part of this term, using equations 84 and 85:

$$\begin{aligned} e^{-in(\theta-\phi)} &= e^{in\phi}\,[\cos\theta - i\sin\theta]^n \\ &= e^{in\phi}\left(\frac{k_x - ik_y}{\kappa}\right)^n. \end{aligned} \tag{91}$$

Our integral is now

$$
\begin{aligned}
H_n^{(1)}(\kappa r) &= \frac{e^{-in\pi/2}}{\pi} \int_{\infty}^{-\infty} \frac{e^{ik_x x + ik_y y}}{-k_y} \left[\frac{(k_x - ik_y)(x+iy)}{\kappa r} \right]^n dk_x \\
&= \frac{e^{-in\pi/2}}{\pi} \int_{-\infty}^{\infty} \frac{e^{ik_x x + ik_y y}}{k_y} \left[\frac{(k_x - ik_y)(x+iy)}{\kappa r} \right]^n dk_x
\end{aligned} \tag{92}
$$

or alternatively

$$
H_n^{(1)}(\kappa r) = \frac{e^{in(\phi - \pi/2)}}{\pi} \int_{-\infty}^{\infty} \frac{e^{ik_x x + ik_y y}}{k_y} \left(\frac{k_x - ik_y}{\kappa} \right)^n dk_x. \tag{93}
$$

Note that this is not really a plane-wave expansion, because of the dependence on ϕ (or equivalently on x and y). The exception to this is when $n = 0$, in which case the representation is very simple:

$$
H_0^{(1)}(\kappa r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{ik_x x + ik_y y}}{k_y} dk_x. \tag{94}
$$

4 Fields from a cylindrical source in terms of plane waves

We can apply the formulae developed in the preceding section in order to expand the fields emitted by our cylindrical sources.

The electrical field emitted by the solenoidal source is

$$
\mathbf{E}(\mathbf{r}) = i\omega A_0 \hat{\boldsymbol{\phi}} e^{ik_z z - i\omega t} H_1^{(1)}(\kappa r). \tag{95}
$$

We are aiming for a plane-wave expansion in Cartesian coordinates. The first step is to write the azimuthal unit vector $\hat{\boldsymbol{\phi}}$ in terms of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$:

$$
\hat{\boldsymbol{\phi}} = (-\sin\phi)\hat{\mathbf{x}} + (\cos\phi)\hat{\mathbf{y}}. \tag{96}
$$

The Cartesian components of the electrical field are therefore

$$
E_x = -i\omega A_0 e^{ik_z z - i\omega t} H_1^{(1)}(\kappa r) \sin\phi \tag{97}
$$

$$
E_y = i\omega A_0 e^{ik_z z - i\omega t} H_1^{(1)}(\kappa r) \cos\phi \tag{98}
$$

$$
E_z = 0. \tag{99}
$$

In the previous section, we developed the expansion of $H_n^{(1)}$ for arbitrary n; when $n = 1$, we have

$$
H_1^{(1)}(\kappa r) = -\frac{ie^{i\phi}}{\pi} \int_{-\infty}^{\infty} \frac{e^{ik_x x + ik_y y}}{k_y} \left(\frac{k_x - ik_y}{\kappa} \right) dk_x. \tag{100}
$$

In order to eliminate ϕ from our formulae, we will also use the expansion for $n = -1$:

$$
\begin{aligned}
H_{-1}^{(1)}(\kappa r) &= \frac{ie^{-i\phi}}{\pi} \int_{-\infty}^{\infty} \frac{e^{ik_x x + ik_y y}}{k_y} \left(\frac{k_x - ik_y}{\kappa} \right)^{-1} dk_x \\
&= \frac{ie^{-i\phi}}{\pi} \int_{-\infty}^{\infty} \frac{e^{ik_x x + ik_y y}}{k_y} \left(\frac{k_x + ik_y}{\kappa} \right) dk_x \qquad (101)
\end{aligned}
$$

where the second line follows because

$$\kappa = \sqrt{k_x^2 + k_y^2}. \qquad (102)$$

A property of Hankel functions is that for integer values of n

$$H_{-n}^{(1)}(z) = (-1)^n H_n^{(1)}(z). \qquad (103)$$

This can be seen for the case $n = 1$ by comparing equations 100 and 101 above.

We start with E_x, and rewrite equation 97 as

$$
\begin{aligned}
E_x &= -i\omega A_0 e^{ik_z z - i\omega t} H_1^{(1)}(\kappa r) \left(\frac{e^{i\phi} - e^{-i\phi}}{2i} \right) \\
&= -\frac{\omega A_0}{2} e^{ik_z z - i\omega t} \left(-H_{-1}^{(1)}(\kappa r) e^{i\phi} - H_1^{(1)}(\kappa r) e^{-i\phi} \right). \qquad (104)
\end{aligned}
$$

By pairing $e^{i\phi}$ with $H_{-1}^{(1)}$ and $e^{-i\phi}$ with $H_1^{(1)}$, we ensure that ϕ will be eliminated when we substitute the expanded versions of the Hankel functions (equations 100 and 101):

$$
\begin{aligned}
E_x &= -\frac{i\omega A_0}{2\pi} e^{ik_z z - i\omega t} \int_{-\infty}^{\infty} \frac{e^{ik_x x + ik_y y}}{k_y} \left[-\left(\frac{k_x + ik_y}{\kappa} \right) + \left(\frac{k_x - ik_y}{\kappa} \right) \right] dk_x \\
&= -\frac{\omega A_0}{\pi\kappa} e^{ik_z z - i\omega t} \int_{-\infty}^{\infty} e^{ik_x x + ik_y y} \, dk_x. \qquad (105)
\end{aligned}
$$

Similarly, we deal with E_y by first rewriting equation 98:

$$
\begin{aligned}
E_y &= i\omega A_0 e^{ik_z z - i\omega t} H_1^{(1)}(\kappa r) \left(\frac{e^{i\phi} + e^{-i\phi}}{2} \right) \\
&= \frac{i\omega A_0}{2} e^{ik_z z - i\omega t} \left(-H_{-1}^{(1)}(\kappa r) e^{i\phi} + H_1^{(1)}(\kappa r) e^{-i\phi} \right). \qquad (106)
\end{aligned}
$$

Substitution then gives

$$
\begin{aligned}
E_y &= -\frac{\omega A_0}{2\pi} e^{ik_z z - i\omega t} \int_{-\infty}^{\infty} \frac{e^{ik_x x + ik_y y}}{k_y} \left[-\left(\frac{k_x + ik_y}{\kappa} \right) - \left(\frac{k_x - ik_y}{\kappa} \right) \right] dk_x \\
&= \frac{\omega A_0}{\pi\kappa} e^{ik_z z - i\omega t} \int_{-\infty}^{\infty} \left(\frac{k_x}{k_y} \right) e^{ik_x x + ik_y y} \, dk_x. \qquad (107)
\end{aligned}
$$

This is the desired result: the Cartesian components of the electric field have been expanded in terms of plane waves.

The same technique can be applied to the magnetic field, which for the solenoidal source is

$$\mathbf{B} = A_0 e^{ik_z z - i\omega t} \left(-\hat{\mathbf{r}} i k_z H_1^{(1)}(\kappa r) + \hat{\mathbf{z}} \kappa H_0^{(1)}(\kappa r) \right). \tag{108}$$

To obtain the Cartesian components, we need the expansion of the radial unit vector:

$$\hat{\mathbf{r}} = (\cos\phi)\hat{\mathbf{x}} + (\sin\phi)\hat{\mathbf{y}}. \tag{109}$$

The Cartesian components of the magnetic field are therefore

$$B_x = -ik_z A_0 e^{ik_z z - i\omega t} H_1^{(1)}(\kappa r)\cos\phi \tag{110}$$

$$B_y = -ik_z A_0 e^{ik_z z - i\omega t} H_1^{(1)}(\kappa r)\sin\phi \tag{111}$$

$$B_z = \kappa A_0 e^{ik_z z - i\omega t} H_0^{(1)}(\kappa r). \tag{112}$$

We can save time by comparing these expressions with E_x and E_y:

$$\begin{aligned} B_x &= -\frac{k_z}{\omega} E_y \\ &= -\frac{k_z A_0}{\pi\kappa} e^{ik_z z - i\omega t} \int_{-\infty}^{\infty} \left(\frac{k_x}{k_y}\right) e^{ik_x x + ik_y y}\, dk_x. \end{aligned} \tag{113}$$

$$\begin{aligned} B_y &= \frac{k_z}{\omega} E_x \\ &= -\frac{k_z A_0}{\pi\kappa} e^{ik_z z - i\omega t} \int_{-\infty}^{\infty} e^{ik_x x + ik_y y}\, dk_x. \end{aligned} \tag{114}$$

There is no ϕ-dependence to worry about in B_z, or in the expansion of $H_0^{(1)}$; we therefore substitute this expansion directly into equation 112 to give

$$B_z = \frac{\kappa A_0}{\pi} e^{ik_z z - i\omega t} \int_{-\infty}^{\infty} \frac{e^{ik_x x + ik_y y}}{k_y}\, dk_x. \tag{115}$$

The waves emitted by the axial current source may be expanded in exactly the same way. However, time can again be saved by noting that the solenoidal and axial fields are related by the transformation indicated in equation 68.

5 Waves impinging on a slab

When dealing with layered systems, it is conventional to take the z-axis as being perpendicular to the layers. In these notes, the z-axis is taken to be along the axis of our cylindrical source. If we are interested in the transmission of

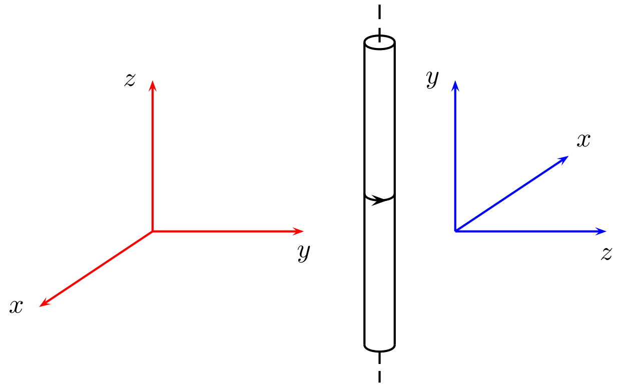

Figure 4: Combining coordinate systems. The coordinates in which the emitted waves were calculated are shown in red; the new coordinates, corresponding to the layered system, are shown in blue. In this case, the current source is aligned with the y-axis in the new coordinates, as indicated by the black cylinder. The region of interest is in the direction of positive z in these coordinates; this means that the old y-axis must coincide with the new z-axis, because the plane-wave expansions are valid only for $y > 0$ (in the old coordinates).

waves from a cylindrical source through a layered system, we must be careful to combine the two coordinate systems correctly.

One combination is illustrated in figure 4; this corresponds to the case when the axis of the cylinder is aligned with the y-axis of the layered system. The diagram illustrates that the transformation

$$\begin{aligned} x &\longleftrightarrow -x \\ y &\longleftrightarrow z \end{aligned} \tag{116}$$

should be applied here.

References

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions.* Dover, New York, 1972.

[2] Wolfgang K. H. Panofsky and Melba Phillips. *Classical Electricity and Magnetism.* Addison-Wesley, Reading, Massachusetts, 1971.