Chapter 4

Properties of Irreducible Representations

Algebra is generous; she often gives more than is asked of her.
—Jean d'Alembert

We have seen in the preceding chapter that a reducible representation can, through a similarity transformation, be brought into block-diagonal form wherein each block is an irreducible representation. Thus, irreducible representations are the basic components from which all representations can be constructed. But the identification of whether a representation is reducible or irreducible is a time-consuming task if it relies solely on methods of linear algebra.[1] In this chapter, we lay the foundation for a more systematic approach to this question by deriving the fundamental theorem of representation theory, called the Great Orthogonality Theorem. The utility of this theorem, and its central role in the applications of group theory to physical problems, stem from the fact that it leads to simple criteria for determining irreducibility and provides a direct way of identifying the number of inequivalent representations for a given group. This theorem is based on two lemmas of Schur, which are the subjects of the first two sections of this chapter.

[1]K. Hoffman and R. Kunze, *Linear Algebra* 2nd edn (Prentice–Hall, Englewood Cliffs, New Jersey, 1971), Ch. 6,7.

4.1 Schur's First Lemma

Schur's two lemmas are concerned with the properties of matrices that commute with all of the matrices of a irreducible representations. The first lemma addresses the properties of matrices which commute with a given *irreducible* representation:

Theorem 4.1 (Schur's First Lemma). A non-zero matrix which commutes with all of the matrices of an irreducible representation is a constant multiple of the unit matrix.

Proof. Let $\{A_1, A_2, \ldots, A_{|G|}\}$ be the matrices of a d-dimensional irreducible representation of a group G, i.e., the A_α are $d \times d$ matrices which cannot all be brought into block-diagonal form by the same similarity transformation. According to Theorem 3.2, we can take these matrices to be unitary without any loss of generality. Suppose there is a matrix M that commutes with all of the A_α:

$$MA_\alpha = A_\alpha M \tag{4.1}$$

for $\alpha = 1, 2, \ldots, |G|$. By taking the adjoint of each of these equations, we obtain

$$A_\alpha^\dagger M^\dagger = M^\dagger A_\alpha^\dagger \,. \tag{4.2}$$

Since the A_α are unitary, $A_\alpha^\dagger = A_\alpha^{-1}$, so multiplying (4.2) from the left and right by A_α yields

$$M^\dagger A_\alpha = A_\alpha M^\dagger \,, \tag{4.3}$$

which demonstrates that, if M commutes with every matrix of a representation, then so does $M^\dagger$. Therefore, given the commutation relations in (4.1) and (4.3) any linear combination of M and $M^\dagger$ also commute with these matrices:

$$(aM + bM^\dagger)A_\alpha = A_\alpha(aM + bM^\dagger) \,,$$

where a and b are any complex constants. In particular, the linear combinations

$$H_1 = M + M^\dagger, \qquad H_2 = i(M - M^\dagger)$$

yield *Hermitian* matrices: $H_i = H_i^\dagger$ for $i = 1, 2$. We will now show that a Hermitian matrix which commutes with all the matrices of an irreducible representation is a constant multiple of the unit matrix. It then follows that M is also such a matrix, since

$$M = \tfrac{1}{2}(H_1 - iH_2) \tag{4.4}$$

The commutation between a general Hermitian matrix H and the A_α is expressed as

$$HA_\alpha = A_\alpha H\,. \tag{4.5}$$

Since H is Hermitian, there is a unitary matrix U which transforms H into a diagonal matrix D (Theorem 3.1):

$$D = U^{-1}HU\,.$$

We now perform the same similarity transformation on (4.5):

$$\begin{aligned} U^{-1}HA_iU &= U^{-1}HUU^{-1}A_iU \\ &= U^{-1}A_iHU = U^{-1}A_iUU^{-1}HU \end{aligned}$$

By defining $\tilde{A}_\alpha = U^{-1}A_\alpha U$, the transformed commutation relation (4.5) reads

$$D\tilde{A}_\alpha = \tilde{A}_\alpha D\,. \tag{4.6}$$

Using the fact that D is a diagonal matrix, i.e., that its matrix elements are $D_{ij} = D_{ii}\delta_{ij}$, where δ_{ij} is the Kronecker delta, the (m, n)th matrix element of the left-hand side of this equation is

$$(D\tilde{A}_\alpha)_{mn} = \sum_k D_{mk}(\tilde{A}_\alpha)_{kn} = \sum_k D_{mm}\delta_{mk}(\tilde{A}_\alpha)_{kn} = D_{mm}(A_\alpha)_{mn}\,.$$

Similarly, the corresponding matrix element on the right-hand side is

$$(\tilde{A}_\alpha D)_{mn} = \sum_k (\tilde{A}_\alpha)_{mk}D_{kn} = \sum_k (\tilde{A}_\alpha)_{mk}D_{nn}\delta_{kn} = (\tilde{A}_\alpha)_{mn}D_{nn}\,.$$

Thus, after a simple rearrangement, the (m, n)th matrix element of (4.6) is

$$(\tilde{A}_\alpha)_{mn}(D_{mm} - D_{nn}) = 0\,. \tag{4.7}$$

There are three cases that we must consider to understand the implications of this equation.

Case I. Suppose that all of the diagonal elements of D are distinct: $D_{mm} \neq D_{nn}$ if $m \neq n$. Then, (4.7) implies that

$$(\tilde{A}_\alpha)_{mn} = 0, \qquad m \neq n\,,$$

i.e., the off-diagonal elements of $\tilde{A}_\alpha$ must vanish, these are *diagonal* matrices and, therefore, according to the discussion in Section 3.3, they form a *reducible* representation composed of d one-dimensional representations. Since the $\tilde{A}_i$ are obtained from the A_i by a similarity transformation, the A_i themselves form a *reducible* representation.

Case II. If all of the diagonal elements of D are equal, i.e. $D_{mm} = D_{nn}$ for *all* m and n, then D is proportional to the unit matrix. The $(\tilde{A}_\alpha)_{mn}$ are not required to vanish for *any* m and n. Thus, only this case is consistent with the requirement that the A_α form an irreducible representation. If D is proportional to the unit matrix, then so is $H = UDU^{-1}$ and, according to (4.4), the matrix M is as well.

Case III. Suppose that the first p diagonal entries of D are equal, but the remaining entries are distinct from these and from each other: $D_{11} = D_{22} = \cdots = D_{pp}$, $D_{mm} \neq D_{nn}$ otherwise. The $(\tilde{A}_\alpha)_{mn}$ must vanish for any pair of unequal diagonal entries. These correspond to the cases where *both* m and n lie in the range $1, 2, \ldots, p$ and where m and n are equal and both greater than p, so *all* the $\tilde{A}_i$ all have the following general form:

$$\tilde{A}_i = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

where B_1 is a $p \times p$ matrix and B_2 is a $(p-d)\times(p-d)$ *diagonal* matrix. Thus, the $\tilde{A}_i$ are block diagonal matrices and are, therefore, *reducible*.

We have shown that if a matrix that not a multiple of the unit matrix commutes with all of the matrices of a representation, then that representation is necessarily *reducible* (Cases I and III). Thus, if a non-zero matrix commutes with all of the matrices of an *irreducible* representation (Case III), that matrix must be a multiple of the unit matrix. This proves Schur's lemma. ∎

4.2 Schur's Second Lemma

Schur's first lemma is concerned with the commutation of a matrix with a given irreducible representation. The second lemma generalizes this to the case of commutation with two distinct irreducible representations which may have different dimensionalities. Its statement is as follows:

Theorem 4.2 (Schur's Second Lemma). Let $\{A_1, A_2, \ldots, A_{|G|}\}$ and $\{A'_1, A'_2, \ldots, A'_{|G|}\}$ be two irreducible representations of a group G of dimensionalities d and d', respectively. If there is a matrix M such that

$$MA_\alpha = A'_\alpha M$$

for $\alpha = 1, 2, \ldots, |G|$, then if $d = d'$, either $M = 0$ or the two representations differ by a similarity transformation. If $d \neq d'$, then $M = 0$.

Proof. Given the commutation relation between M and the two irreducible representations,

$$MA_\alpha = A'_\alpha M \,, \tag{4.8}$$

we begin by taking the adjoint:

$$A^\dagger_\alpha M^\dagger = M^\dagger A'^\dagger_\alpha \,. \tag{4.9}$$

Since, according to Theorem 3.2, the A_α may be assumed to be unitary, $A^\dagger_\alpha = A^{-1}_\alpha$, so (4.9) becomes

$$A^{-1}_\alpha M^\dagger = M^\dagger A'^{-1}_\alpha \,. \tag{4.10}$$

By multiplying this equation from the left by M,

$$MA^{-1}_\alpha M^\dagger = MM^\dagger A'^{-1}_\alpha \,,$$

and utilizing the commutation relation (4.8) to write

$$MA^{-1}_\alpha = A'^{-1}_\alpha M \,,$$

we obtain

$$A'^{-1}_\alpha MM^\dagger = MM^\dagger A'^{-1}_\alpha \,.$$

Thus, the $d' \times d'$ matrix $MM^\dagger$ commutes with all the matrices of an irreducible representation. According to Schur's First Lemma, $MM^\dagger$ must therefore be a constant multiple of the unit matrix,

$$MM^\dagger = cI\,, \tag{4.11}$$

where c is a constant. We now consider individual cases.

Case I. $d = d'$. If $c \neq 0$, Eq. (4.11) implies that[2]

$$M^{-1} = \frac{1}{c} M^\dagger\,.$$

Thus, we can rearrange (4.8) as

$$A_\alpha = M^{-1} A'_\alpha M\,,$$

so our two representations are related by a similarity transformation and are, therefore, equivalent.

If $c = 0$, then $MM^\dagger = 0$. The (i,j)th matrix element of this product is

$$(MM^\dagger)_{ij} = \sum_k M_{ik}(M^\dagger)_{kj} = \sum_k M_{ik}M^*_{jk} = 0\,.$$

By setting $i = j$, we obtain

$$\sum_k M_{ik}M^*_{ik} = \sum_k |M_{ik}|^2 = 0\,,$$

which implies that $M_{ik} = 0$ for *all* i and k, i.e., that M is the zero matrix. This completes the first part of the proof.

Case II. $d \neq d'$. We take $d < d'$. Then M is a *rectangular* matrix with d columns and d' rows:

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1d} \\ M_{21} & \cdots & M_{2d} \\ \vdots & \ddots & \vdots \\ M_{d'1} & \cdots & M_{d'd} \end{pmatrix}.$$

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[2]By multiplying (4.10) from the *right* by M and following analogous steps as above, one can show that $M^\dagger M = cI$, so that the matrix $c^{-1}M^\dagger$ is both the left and right inverse of M.

We can make a $d' \times d'$ matrix N from M by adding $d' - d$ columns of zeros:

$$N = \begin{pmatrix} M_{11} & \cdots & M_{1d} & 0 & \cdots & 0 \\ M_{21} & \cdots & M_{2d} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_{d'1} & \cdots & M_{d'd} & 0 & \cdots & 0 \end{pmatrix} \equiv (M, 0)\,.$$

Taking the adjoint of this matrix yields

$$N^\dagger = \begin{pmatrix} M_{11} & M_{21}^* & \cdots & M_{d'1}^* \\ M_{12}^* & M_{22}^* & \cdots & M_{d'2}^* \\ \vdots & \vdots & \ddots & \vdots \\ M_{1d}^* & M_{2d}^* & \cdots & M_{d'd}^* \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} M^\dagger \\ 0 \end{pmatrix}.$$

Note that this construction maintains the product $MM^\dagger$:

$$NN^\dagger = (M, 0) \begin{pmatrix} M^\dagger \\ 0 \end{pmatrix} = MM^\dagger = cI\,.$$

The determinant of N is clearly zero. Thus,

$$\det(NN^\dagger) = \det(N)\det(N^\dagger) = c^{d'} = 0$$

so $c = 0$, which means that $MM^\dagger = 0$. Proceeding as in Case I, we conclude that this implies $M = 0$. This completes the second part of the proof. ▮

4.3 The Great Orthogonality Theorem

Schur's lemmas provide restrictions on the form of matrices which commute with all of the matrices of irreducible representations. But the group property enables the construction of many matrices which satisfy the relations in Schur's First and Second Lemmas. The interplay between these two facts provides the basis for proving the Great Orthogonality Theorem. The statement of this theorem is as follows:

Theorem 4.3 (Great Orthogonality Theorem). Let $\{A_1, A_2, \ldots, A_{|G|}\}$ and $\{A'_1, A'_2, \ldots, A'_{|G|}\}$ be two inequivalent irreducible representations of a group G with elements $\{g_1, g_2, \ldots, g_{|G|}\}$ and which have dimensionalities d and d', respectively. The matrices A_α and A'_α in the two representations correspond to the element g_α in G. Then

$$\sum_\alpha (A_\alpha)^*_{ij}(A'_\alpha)_{i'j'} = 0$$

for all matrix elements. For the elements of a single unitary irreducible representation, we have

$$\sum_\alpha (A_\alpha)^*_{ij}(A_\alpha)_{i'j'} = \frac{|G|}{d}\delta_{i,i'}\delta_{j,j'}\,,$$

where d is the dimension of the representation.

Proof. Consider the matrix

$$M = \sum_\alpha A'_\alpha X A_\alpha^{-1}\,, \tag{4.12}$$

where X is an arbitrary matrix with d' rows and d columns, so that M is a $d' \times d'$ matrix. We will show that for any matrix X, M satisfies a commutation relation of the type discussed in Schur's Lemmas.

We now multiply M from the left by the matrix A'_β corresponding to some matrix in the the "primed" representation:

$$\begin{aligned} A'_\beta M &= \sum_\alpha A'_\beta A'_\alpha X A_\alpha^{-1} \\ &= \sum_\alpha A'_\beta A'_\alpha X A_\alpha^{-1} A_\beta^{-1} A_\beta \\ &= \sum_\alpha A'_\beta A'_\alpha X (A_\beta A_\alpha)^{-1} A_\beta\,. \end{aligned} \tag{4.13}$$

Since the A_α and A'_α form representations of G, the products $A_\alpha A_\beta$ and $A'_\alpha A'_\beta$ yield matrices A_γ and A'_γ, respectively, both corresponding to the same element in G because representations preserve the group composition rule. Hence, by the Rearrangement Theorem (Theorem 2.1), we can write the summation over α on the right-hand side of this equation as

$$\sum_\alpha A'_\beta A'_\alpha X(A_\beta A_\alpha)^{-1} = \sum_\gamma A'_\gamma X A_\gamma^{-1} = M\,.$$

Substituting this result into (4.13) yields

$$A'_\beta M = M A_\beta\,. \tag{4.14}$$

Depending on the nature of the two representations, this is precisely the situation addressed by Schur's First and Second Lemmas. We consider the cases of equivalent and inequivalent representations separately.

Case I. $d \neq d'$ or, if $d = d'$, the representations are inequivalent (i.e., not related by a similarity transformation). Schur's Second Lemma then implies that M must be the zero matrix, i.e., that each matrix element of M is zero. From the definition (4.12), we see that this requires

$$M_{ii'} = \sum_\alpha \sum_{jj'} (A'_\alpha)_{ij} X_{jj'} (A_\alpha^{-1})_{j'i'} = 0\,. \tag{4.15}$$

By writing this sum as (note that because all sums are finite, their order can be changed at will)

$$\sum_{jj'} X_{jj'} \Big[\sum_\alpha (A_\alpha)'_{ij} (A_\alpha^{-1})_{j'i'} \Big] = 0\,, \tag{4.16}$$

we see that, since X is arbitrary, each of its entries may be varied arbitrarily and independently without affecting the vanishing of the sum. The only way to ensure this is to require that the coefficients of the $X_{jj'}$ vanish:

$$\sum_\alpha (A'_\alpha)_{ij} (A_\alpha^{-1})_{j'i'} = 0\,.$$

For unitary representations, $(A_\alpha^{-1})_{j'i'} = (A_\alpha)^*_{i'j'}$, so this equation reduces to

$$\sum_\alpha (A'_\alpha)_{ij}(A_\alpha)^*_{i'j'} = 0\,,$$

which proves the first part of the theorem.

Case II. $d = d'$ and the representations are equivalent. According to Schur's First Lemma, $M = cI$, so,

$$cI = \sum_\alpha A_\alpha X A_\alpha^{-1}\,. \tag{4.17}$$

Taking the trace of both sides of this equation,

$$\underbrace{\mathrm{tr}(cI)}_{cd} = \mathrm{tr}\Big(\sum_\alpha A_\alpha X A_\alpha^{-1}\Big) = \sum_\alpha \mathrm{tr}(A_\alpha X A_\alpha^{-1}) = \underbrace{\sum_\alpha \mathrm{tr}(X)}_{|G|\,\mathrm{tr}(X)}\,,$$

yields an expression for c:

$$c = \frac{|G|}{d}\,\mathrm{tr}(X)\,.$$

Substituting this into Eq. (4.17) and expressing the resulting equation in terms of matrix elements, yields

$$\sum_{jj'} X_{jj'}\Big[\sum_\alpha (A_\alpha)_{ij}(A_\alpha^{-1})_{j'i'}\Big] = \frac{|G|}{d}\delta_{i,i'}\sum_j X_{jj}\,,$$

or, after a simple rearrangement,

$$\sum_{jj'} X_{jj'}\Big[\sum_\alpha (A_\alpha)_{ij}(A_\alpha^{-1})_{j'i'} - \frac{|G|}{d}\delta_{i,i'}\delta_{j,j'}\Big] = 0\,.$$

This equation must remain valid under any independent variation of the matrix elements of X. Thus, we must require that the coefficient of $X_{jj'}$ vanishes identically:

$$\sum_\alpha (A_\alpha)_{ij}(A_\alpha^{-1})_{j'i'} = \frac{|G|}{d}\delta_{i,i'}\delta_{j,j'}\,.$$

Since the representation is unitary, this is equivalent to

$$\sum_\alpha (A_\alpha)_{ij}(A_\alpha)^*_{i'j'} = \frac{|G|}{d}\delta_{i,i'}\delta_{j,j'}\,.$$

This proves the second part of the theorem. ▮

4.4 Some Immediate Consequences of the Great Orthogonality Theorem

The Great Orthogonality Theorem establishes a relation between matrix elements of the irreducible representations of a group. Suppose we denote the αth matrix in the kth irreducible representation by A^k_α and the (i,j)th element of this matrix by $(A^k_\alpha)_{ij}$. We can then combine the two statements of the Great Orthogonality Theorem as

$$\sum_\alpha (A^k_\alpha)_{ij}(A^{k'}_\alpha)^*_{i'j'} = \frac{|G|}{d}\delta_{i,i'}\delta_{j,j'}\delta_{k,k'} \tag{4.18}$$

This expression helps us to understand the motivation for the name "Orthogonality Theorem" by inviting us to consider the matrix elements of irreducible representations as entries in $|G|$-component vectors, i.e., vectors in a space of dimensionality $|G|$:

$$\boldsymbol{V}^k_{ij} = \Big[(A^k_1)_{ij}, (A^k_2)_{ij}, \ldots, (A^k_{|G|})_{ij}\Big]$$

According to the statement of the Great Orthogonality Theorem, two such vectors are orthogonal if they differ in any one of the indices i, j, or k, since (4.18) requires that

$$\boldsymbol{V}^k_{ij}\cdot\boldsymbol{V}^{k'}_{i'j'} = \frac{|G|}{d}\delta_{i,i'}\delta_{j,j'}\delta_{k,k'}$$

But, in a $|G|$-dimensional space there are at most $|G|$ mutually orthogonal vectors. To see the consequences of this, suppose we have irreducible representations of dimensionalities $d_1, d_2, \ldots$, where the d_k

are positive integers. For the k representations, there are d_k choices for each of i and j, i.e., there are d_k^2 matrix elements in each matrix of the representation. Summing over all irreducible representations, we obtain the inequality

$$\sum_k d_k^2 \leq |G| \tag{4.19}$$

Thus, the order of the group acts as an upper bound both for the number and the dimensionalities of the irreducible representations. In particular, a finite group can have only a *finite* number of irreducible representations. We will see later that the *equality* in (4.19) always holds.

Example 4.1. For the group S_3, we have that $|G| = 6$ and we have already identified two one-dimensional irreducible representations and one two-dimensional irreducible representation (Example 3.2). Thus, using (4.19), we have

$$\sum_k d_k^2 = 1^2 + 1^2 + 2^2 = 6$$

so the Great Orthogonality Theorem tells us that there are no additional distinct irreducible representations.

For the two element group, we have found two one-dimensional representations, $\{1, 1\}$ and $\{1, -1\}$ (Example 3.3). According to the inequality in (4.19),

$$\sum_k d_k^2 = 1 + 1 = 2$$

so these are the only two irreducible representations of this group. ∎

4.5 Summary

The central result of this chapter is the statement and proof of the Great Orthogonality Theorem. Essentially all of the applications in the next several chapters are consequences of this theorem. The important advance provided this theorem is that it provides an orthogonality relation between the entries of the matrices of the irreducible representations of a group. While this can be used to test whether a given representation is reducible or irreducible (Problem Set 6), its main role will be in a somewhat "reduced" form, such as that used in Sec. 4.4 to place bounds on the number of irreducible representations of a finite group. One of the most important aspects of the Great Orthogonality Theorem for applications to physical problems is in the construction of "character tables," i.e., tables of traces of matrices of an irreducible representation. This is taken up in the next chapter.