Chapter 5

Characters and Character Tables

In great mathematics there is a very high degree of unexpectedness, combined with inevitability and economy.

—Godfrey H. Hardy[1]

In the preceding chapter, we proved the Great Orthogonality Theorem, which is a statement about the orthogonality between the matrix elements corresponding to different irreducible representations of a group. For many applications of group theory, however, the full matrix representations of a group are not required, but only the traces within classes of group elements—called "characters." A typical application involves determining whether a given representation is reducible or irreducible and, if it is reducible, to identify the irreducible representations contained within that representation.

In this chapter, we develop the mathematical machinery that is used to assemble the characters of the irreducible representations of a group in what are called "character tables." The compilation of character tables requires two types of input: the order of the group and the number of classes it contains. These quantities provide stringent restrictions

[1] G.H. Hardy, *A Mathematician's Apology* (Cambridge University Press, London, 1941)

on the number of irreducible representations and their dimensionalities. Moreover, orthogonality relations derived from the Great Orthogonality Theorem will be shown to provide constraints on characters of different irreducible representations, which considerably simplifies the construction of character tables.

5.1 Orthogonality Relations

The Great Orthogonality Theorem,

$$\sum_\alpha (A_\alpha^k)_{ij}(A_\alpha^{k'})^*_{i'j'} = \frac{|G|}{d_k}\delta_{i,i'}\delta_{j,j'}\delta_{k,k'} \tag{5.1}$$

is a relationship between the matrix elements of the irreducible representations of a group G. In this section, we show how this statement can be manipulated into an expression solely in terms of the traces of the matrices in these representations. This will open the way to establishing a sum rule between the number of irreducible representations and the number of classes in a group.

We begin by setting $j = i$ and $j' = i'$ in (5.1),

$$\sum_\alpha (A_\alpha^k)_{ii}(A_\alpha^{k'})^*_{i'i'} = \frac{|G|}{d_k}\delta_{i,i'}\delta_{k,k'}\,, \tag{5.2}$$

where we have used the fact that $\delta_{i,i'}\delta_{i,i'} = \delta_{i,i'}$. Summing over i and i' on the left-hand side of this equation yields

$$\begin{aligned}\sum_{i,i'}\sum_\alpha (A_\alpha^k)_{ii}(A_\alpha^{k'})_{i'i'} &= \sum_\alpha \underbrace{\Bigg[\sum_i (A_\alpha^k)_{ii}\Bigg]}_{\mathrm{tr}(A_\alpha^k)}\underbrace{\Bigg[\sum_{i'}(A_\alpha^{k'})^*_{i'i'}\Bigg]}_{\mathrm{tr}(A_\alpha^{k'})^*} \\ &= \sum_\alpha \mathrm{tr}(A_\alpha^k)\mathrm{tr}(A_\alpha^{k'})^*\,,\end{aligned}$$

and, by summing over i and i' on the right-hand side of (5.2), we obtain

$$\frac{|G|}{d_k}\delta_{k,k'}\sum_i\sum_{i'}\delta_{i,i'} = \frac{|G|}{d_k}\delta_{k,k'}\underbrace{\sum_i 1}_{d_k} = |G|\delta_{k,k'}\,.$$

We have thereby reduced the Great Orthogonality Theorem to

$$\sum_\alpha \mathrm{tr}(A_\alpha^k)\mathrm{tr}(A_\alpha^{k'*}) = |G|\delta_{k,k'}\,. \tag{5.3}$$

This expression can be written in a more useful form by observing that matrices corresponding to elements in the same conjugacy class have the same trace. To see this, recall the definition in Section 2.6 of the conjugacy of two elements a and b group G. There must be an element g in G such that $a = gbg^{-1}$. Any representation $\{A_\alpha\}$, reducible or irreducible, must preserve this relation:

$$A_a = A_g A_b A_{g^{-1}}\,.$$

This representation must also have the property that $A_{g^{-1}} = A_g^{-1}$. Thus (Problem 2, Problem Set 4),

$$\mathrm{tr}(A_a) = \mathrm{tr}(A_g A_b A_g^{-1}) = \mathrm{tr}(A_g^{-1} A_g A_b) = \mathrm{tr}(A_b)\,.$$

We can now introduce the notation χ_α^k for the trace corresponding to *all* of the elements of the αth class of the kth irreducible representation. This is called the **character** of the class. If there are n_α elements in this class, then we can write the relation (5.3) in terms of characters as a sum over conjugacy classes

$$\sum_{\alpha=1}^{\mathcal{C}} n_\alpha \chi_\alpha^k \chi_\alpha^{k'*} = |G|\delta_{k,k'}\,, \tag{5.4}$$

where $\mathcal{C}$ is the number of conjugacy classes. In arriving at this relation, we have proven the following theorem:

Theorem 5.1 (Orthogonality Theorem for Characters). The characters of the irreducible representations of a group obey the relation

$$\sum_\alpha n_\alpha \chi_\alpha^k \chi_\alpha^{k'*} = |G|\delta_{k,k'}\,.$$

This orthogonality theorem can be used to deduce a relationship between the number classes of a group and the number of irreducible representations. By rearranging (5.4) as

$$\sum_\alpha \left[\left(\frac{n_\alpha}{|G|}\right)^{1/2}\chi_\alpha^k\right]\left[\left(\frac{n_\alpha}{|G|}\right)^{1/2}\chi_\alpha^{k'*}\right] = \delta_{k,k'}$$

and introducing the vectors

$$\widetilde{\boldsymbol{\chi}}^k = |G|^{-1/2}(\sqrt{n_1}\chi_1^k, \sqrt{n_2}\chi_2^k, \ldots, \sqrt{n_{\mathcal{C}}}\chi_{\mathcal{C}}^k)\,,$$

we can write the orthogonality relation for characters as

$$\widetilde{\boldsymbol{\chi}}^k \cdot \widetilde{\boldsymbol{\chi}}^{k'} = \delta_{k,k'}\,.$$

The $\widetilde{\boldsymbol{\chi}}^k$ reside in a space whose dimension is the number of classes $\mathcal{C}$ in the group. Thus, the maximum number of a set of mutually orthogonal vectors in this space is $\mathcal{C}$. But these vectors are labelled by an index k corresponding to the irreducible representations of the group. Hence, *the number of irreducible representations must be less than or equal to the number of classes.*

It is also possible[2] to obtain an orthogonality relation with the roles of the irreducible representations and classes reversed in comparison to that in Theorem 5.1:

$$\sum_k \chi_\alpha^k \chi_\beta^{k*} = \frac{|G|}{n_\alpha}\delta_{\alpha,\beta}\,. \tag{5.5}$$

By following analogous reasoning as above, we can deduce that this orthogonality relation implies that *the number of irreducible representations must be greater than or equal to the number of classes.* Combined with the statement of Theorem 5.1, we have the following theorem:

Theorem 5.2. The number of irreducible representations of a group is equal to the number of conjugacy classes of that group.

Example 5.1. For Abelian subgroups each element is in a class by itself (Problem 6, Problem Set 3). Thus, the number of classes is equal to the order of the group, so, according to Theorem 5.2, the number of irreducible representations must also equal the order of the group. When combined with the restriction imposed by Eqn. (4.19), which we can now write as

$$\sum_{k=1}^{|G|} d_k^2 = |G|\,,$$

[2] M. Hamermesh, *Group Theory and its Application to Physical Problems* (Dover, 1989, New York) pp. 106–110.

we have an alternative way (cf. Problem 4, Problem Set 5) of seeing that all of the the irreducible representations of an Abelian group are one-dimensional, i.e., $d_k = 1$, for $k = 1, 2, \ldots, |G|$. ∎

Example 5.2. For the group S_3, there are three classes: $\{e\}$, $\{a, b, c\}$, and $\{d, f\}$ (Example 2.9). Thus, there are three irreducible representations which, as we have seen, consist of two one-dimensional representations and one two-dimensional representation. ∎

5.2 The Decomposition Theorem

One of the main uses of characters is in the decomposition of a given reducible representation into its constituent irreducible representations. The procedure by which this is accomplished is analogous to projecting a vector onto a set of complete orthogonal basis vectors. The theorem which provides the foundation for carrying this out with characters is the following:

Theorem 5.3 (Decomposition Theorem). The character χ_α for the αth class of any representation can be written uniquely in terms of the corresponding characters of the irreducible representations of the group as

$$\chi_\alpha = \sum_k a_k \chi_\alpha^k \,,$$

where

$$a_k = \frac{1}{|G|} \sum_\alpha n_\alpha \chi_\alpha^{k*} \chi_\alpha \,.$$

Proof. For a reducible representation, the same similarity transformation brings all of the matrices into the same block-diagonal form. In this form, the matrix A_α can be written as the direct sum of matrices A_j^k of irreducible representations:

$$A_\alpha = A_\alpha^{k_1} \oplus A_\alpha^{k_2} \oplus \cdots \oplus A_\alpha^{k_n} \,,$$

where $\alpha = 1, 2, \ldots, |G|$ and $k_1, k_2, \ldots k_n$ label irreducible representations. Given this, and the fact that similarity transformations leave the trace invariant, we can write the character χ_i of this reducible representation corresponding to the ith class as

$$\chi_\alpha = \sum_k a_k \chi_\alpha^k \,, \tag{5.6}$$

where the a_k must be *nonnegative integers*. We now multiply both sides of this equation by $n_\alpha \chi_\alpha^{k'*}$, sum over α, and use the orthogonality relation (5.4):

$$\sum_\alpha n_\alpha \chi_\alpha^{k'*} \chi_\alpha = \sum_k a_k \underbrace{\sum_\alpha n_\alpha \chi_\alpha^k \chi_\alpha^{k'*}}_{|G|\delta_{k,k'}} = |G| a_{k'}$$

Thus,

$$a_{k'} = \frac{1}{|G|} \sum_\alpha n_\alpha \chi_\alpha^{k'*} \chi_\alpha \,, \tag{5.7}$$

so $a_{k'}$ is the projection of the reducible representation onto the k'th irreducible representation. Note that, because the number of irreducible representations equals the number of classes, the orthogonal vectors of characters span the space whose dimensionality is the number of classes, so this decomposition is unique. ∎

The Decomposition Theorem reduces the task of determining the irreducible representations contained within a reducible representation to one of vector algebra. Unless a particular application requires the matrix forms of the representations, there is no need to block-diagonalize a representation to identify its irreducible components.

We can follow a procedure analogous to that used to prove the Decomposition Theorem to derive a simple criterion to identify whether a representation is reducible or irreducible. We begin with the decomposition (5.6) and take its complex conjugate:

$$\chi_\alpha^* = \sum_{k'} a_{k'} \chi_\alpha^{k'*} \,, \tag{5.8}$$

where we have used the fact that the a_k are integers, so $a_k^* = a_k$. We now take the product of (5.6) and (5.8), multiply by n_α, sum over α, and invoke (5.4):

$$\sum_\alpha n_\alpha \chi_\alpha \chi_\alpha^* = \sum_{k,k'} a_k a_{k'} \underbrace{\sum_i n_\alpha \chi_\alpha^k \chi_\alpha^{k'*}}_{|G|\delta_{k,k'}} = |G| \sum_k a_k^2 .$$

Thus,

$$\sum_\alpha n_\alpha |\chi_\alpha^2| = |G| \sum_k a_k^2 . \tag{5.9}$$

If the representation in question is irreducible, then all of the a_k are zero, except for the one corresponding to that irreducible representation, which is equal to unity. If the representation is reducible, then there will be at least two of the a_k which are positive integers. We can summarize these observations with a simple criterion for reducibility. If the representation is irreducible, then

$$\sum_\alpha n_\alpha |\chi_\alpha|^2 = |G| , \tag{5.10}$$

and if the representation is reducible,

$$\sum_\alpha n_\alpha |\chi_\alpha|^2 > |G| . \tag{5.11}$$

Example 5.3. Consider the following representation of S_3:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = \tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad b = \tfrac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

$$c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \qquad f = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

There are three classes of this group, $\{e\}$, $\{a, b, c\}$, and $\{d, f\}$, so we have $n_1 = 1$, $n_2 = 3$, and $n_3 = 2$, respectively. The corresponding characters are

$$\chi_1 = 2, \qquad \chi_2 = 0, \qquad \chi_3 = -1 .$$

Forming the sum in (5.9), we obtain

$$\sum_{\alpha=1}^{3} n_\alpha |\chi_\alpha|^2 = (1\times 4) + (3\times 0) + (2\times 1) = 6\,,$$

which is equal to the order of the group. Therefore, this representation is *irreducible*, as we have already demonstrated in Example 3.4 and in Problem 1, Problem Set 6.

Example 5.4. Another representation of S_3 is

$$e=d=f=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a=b=c=\tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

The characters corresponding to the three classes are now

$$\chi_1 = 2, \qquad \chi_2 = 0, \qquad \chi_3 = 2\,.$$

Forming the sum in (5.9), we find

$$\sum_{i=1}^{3} n_1 |\chi_i|^2 = (1\times 4) + (3\times 0) + (2\times 4) = 12\,,$$

which is *greater* than the order of the group, so this representation is *reducible* (cf. Problem 2, Problem Set 6). To determine the irreducible constituents of this representation, we use the decomposition theorem. There are three irreducible representations of S_3: the one-dimensional identical representation, with characters

$$\chi_1^1 = 1, \qquad \chi_2^1 = 1, \qquad \chi_3^1 = 1\,,$$

the one-dimensional “parity” representation, with characters

$$\chi_1^2 = 1, \qquad \chi_2^2 = -1, \qquad \chi_3^2 = 1\,,$$

and the two-dimensional “coordinate” representation discussed above in Example 5.3, with characters

$$\chi_1^3 = 2, \qquad \chi_2^3 = 0, \qquad \chi_3^3 = -1\,.$$

We now calculate the a_k using the expression in Equation (5.7). These determine the "projections" of the characters of the reducible representation onto the characters of the irreducible representation. We obtain

$$a_1 = \tfrac{1}{6}\Big[(1\times 1\times 2) + (3\times 1\times 0) + (2\times 1\times 2)\Big] = 1\,,$$

$$a_2 = \tfrac{1}{6}\Big[(1\times 1\times 2) + (3\times -1\times 0) + (2\times 1\times 2)\Big] = 1\,,$$

$$a_3 = \tfrac{1}{6}\Big[(1\times 2\times 2) + (3\times 0\times 0) + (2\times -1\times 2)\Big] = 0\,.$$

Thus, this reducible representation is composed of the identical representation and the "parity" representation, with no contribution from the "coordinate" representation. The block-diagonal form of this representation is, therefore,

$$e = d = f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = b = c = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which is the result obtained in Problem 5, Problem Set 5 by applying matrix methods. ∎

5.3 The Regular Representation

Our construction of irreducible representations has thus far proceeded in an essentially *ad hoc* fashion, relying in large part on physical arguments. We have not yet developed a systematic procedure for constructing all of the irreducible representations of a group. In this section, we introduce a method, based on what is called the "regular" representation, which enables us to accomplish this. However, our purpose for introducing such a methodology is not the determination of irreducible representations as such, since even for relatively simple groups, the approach we describe would present a computationally demanding process, but as a theoretical tool for proving a theorem. Moreover, we will find that, for applications of group theory to quantum mechanics, the irreducible representations of the group of operations that leave Hamiltonian invariant will emerge naturally without having to rely on any auxiliary constructions.

The **regular representation** is a reducible representation that is obtained directly from the multiplication table of a group. As we will show below, this representation contains every irreducible representation of a group at least once. The construction of the regular representation is based on arranging the multiplication table of a group so that the unit element appears along the main diagonal of the table. Within such an arrangement the columns (or rows) of the table are labelled by the group elements, arranged in any order, and the corresponding order of the inverses labels the rows (or columns).

As an example, consider the multiplication table for S_3 (Section 2.4) arranged in the way just described:

	e	a	b	c	d	f
$e=e^{-1}$	e	a	b	c	d	f
$a=a^{-1}$	a	e	d	f	b	c
$b=b^{-1}$	b	f	e	d	c	a
$c=c^{-1}$	c	d	f	e	a	b
$f=d^{-1}$	f	b	c	a	e	d
$d=f^{-1}$	d	c	a	b	f	e

The matrices of the regular representation are obtained by regarding the multiplication table as an $|G|\times|G|$ array from which the matrix representation for each group element is assembled by putting a '1' where that element appears in the multiplication table and zero elsewhere. For example, the matrices corresponding to the unit e and the element a are

$$e\to\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix},\qquad a\to\begin{pmatrix}0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\\0&0&1&0&0&0\end{pmatrix}$$

with analogous matrices for the other group elements.

Our first task is to show that the regular 'representation' is indeed a representation of the group. First of all, it is clear that the mapping we have described is one-to-one. For any two elements g_1 and g_2 of this group, we denote the matrices in the regular representation that correspond to these elements as $A_{\rm reg}(g_1)$ and $A_{\rm reg}(g_2)$. Thus, to show that these matrices form a representation of S_3, we need to verify that

$$A_{\rm reg}(g_1 g_2) = A_{\rm reg}(g_1) A_{\rm reg}(g_2)\,,$$

i.e., that the multiplication table is preserved by this representation. We consider this relation expressed in terms of matrix elements:

$$\Big[A_{\rm reg}(g_1 g_2)\Big]_{ij} = \sum_k \Big[A_{\rm reg}(g_1)\Big]_{ik} \Big[A_{\rm reg}(g_2)\Big]_{kj}\,. \tag{5.12}$$

From the way the regular representation has been constructed, the ith row index of these matrix elements can be labelled the inverse of the ith group element g_i^{-1} and the jth column can be labelled by the jth group element g_j:

$$\Big[A_{\rm reg}(g_1 g_2)\Big]_{ij} = \Big[A_{\rm reg}(g_1 g_2)\Big]_{g_i^{-1}, g_j} = \begin{cases} 1, & \text{if } g_i^{-1} g_j = g_1 g_2; \\ 0; & \text{otherwise} \end{cases}$$

$$\Big[A_{\rm reg}(g_1)\Big]_{ik} = \Big[A_{\rm reg}(g_1)\Big]_{g_i^{-1}, g_k} = \begin{cases} 1, & \text{if } g_i^{-1} g_k = g_1; \\ 0; & \text{otherwise} \end{cases}$$

$$\Big[A_{\rm reg}(g_2)\Big]_{kj} = \Big[A_{\rm reg}(g_2)\Big]_{g_k^{-1}, g_j} = \begin{cases} 1, & \text{if } g_k^{-1} g_j = g_2; \\ 0; & \text{otherwise} \end{cases}$$

Therefore, in the sum over k in (5.12), we have nonzero entries only when

$$g_1 g_2 = (g_i^{-1} g_k)(g_k^{-1} g_j) = g_i^{-1} g_j\,,$$

which gives precisely the nonzero matrix elements of $A_{\rm reg}(g_1 g_2)$. Hence, the matrices $A_{\rm reg}(g_1)$ preserve the group multiplication table and thereby form a faithful representation of the group.

Our main purpose in introducing the regular representation is to prove the following theorem:

Theorem 5.4. The dimensionalities d_k of the irreducible representations of a group are related to the order $|G|$ of the group by

$$\sum_k d_k^2 = |G| \,.$$

This theorem shows that the inequality (4.19), which was deduced directly from the Great Orthogonality Theorem is, in fact, an equality.

Proof. We first show, using Eqn. (5.9), that the regular representation is reducible. To evaluate the sums on the left-hand side of this equation, we note that, from the construction of the regular representation, the characters $\chi_{\mathrm{reg},i}$ vanish for every class except for that corresponding to the unit element. Denoting this character by $\chi_{\mathrm{reg},e}$, we see that its value must be equal to the order of the group:

$$\chi_{\mathrm{reg},e} = |G| \,.$$

Thus,

$$\sum_\alpha n_\alpha |\chi_\alpha|^2 = \chi_{\mathrm{reg},e}^2 = |G|^2 \,,$$

which, for $|G| > 1$ is *greater* than $|G|$. Thus, for groups other than the single-element group $\{e\}$, the regular representation is reducible.

We will now use the Decomposition Theorem to identify the irreducible constituents of the regular representation. Thus, the characters $\chi_{\mathrm{reg},\alpha}$ for the αth class in the regular representation can be written as

$$\chi_{\mathrm{reg},\alpha} = \sum_k a_k \chi_\alpha^k \,.$$

According to the Decomposition Theorem, the a_k are given by

$$a_k = \frac{1}{|G|} \sum_\alpha n_\alpha \chi_\alpha^{k*} \chi_{\mathrm{reg},\alpha} \,.$$

We again use the fact that $\chi_{\mathrm{reg},e} = |G|$, with all other characters vanishing. The corresponding value of χ_e^k is determined by taking the trace of the identity matrix whose dimensionality is that of the kth irreducible representation: $\chi_e^k = d_k$. Therefore, the Decomposition Theorem yields

$$a_k = \frac{1}{|G|} \times d_k \times |G| = d_k\,,$$

i.e., the kth irreducible representation appears d_k times in the regular representation: each one-dimensional irreducible representation appears once, each two-dimensional irreducible representation appears twice, and so on. Since the dimensionality of the regular representation is $|G|$, and since a_k is the number of times the kth irreducible representation appears in the regular representation, we have the constraint

$$\sum_k a_k d_k = |G|\,,$$

i.e.,

$$\sum_k d_k^2 = |G|\,.$$

∎

This sum rule, and that equating the number of classes to the number of irreducible representations (Theorem 5.2), relate a property of the abstract group (its order and the number of classes) to a property of the irreducible representations (their number and dimensionality). The application of these rules and the orthogonality theorems for characters is the basis for constructing character tables. This is described in the next section.

5.4 Character Tables

Character tables are central to many applications of group theory to physical problems, especially those involving the decomposition of reducible representations into their irreducible components. Many textbooks on group theory contain compilations of character tables for the most common groups. In this section, we will describe the construction of character tables for S_3. We will utilize two types of information: sum rules for the number and dimensionalities of the irreducible representations, and orthogonality relations for the characters. Additionally, the group multiplication table can be used to establish relationships for one-dimensional representations. By convention, characters tables are displayed with the columns labelled by the classes and the rows by the irreducible representations.

The first step in the construction of this character table is to note that, since $|S_3| = 6$ and there are three classes (Example 2.9), there are 3 irreducible representations whose dimensionalities must satisfy

$$d_1^2 + d_2^2 + d_3^2 = 6\,.$$

The unique solution of this equation (with only positive integers) is $d_1 = 1$, $d_2 = 1$, and $d_3 = 2$, so there are two one-dimensional irreducible representations and one two-dimensional irreducible representation.

In the character table for any group, several entries can be made immediately. The identical representation, where all elements are equal to unity, is always a one-dimensional irreducible representation. Similarly, the characters corresponding to the unit element are equal to the dimensionality of that representation, since they are calculated from the trace of the identity matrix with that dimensionality. Thus, denoting by α, β, γ, and δ quantities that are to be determined, the character table for S_3 is:

S_3	$\{e\}$	$\{a,b,c\}$	$\{d,f\}$
Γ_1	1	1	1
Γ_2	1	α	β
Γ_3	2	γ	δ

where the Γ_i are a standard label for the irreducible representations.

The remaining entries are determined from the orthogonality relations for characters and, for one-dimensional irreducible representations, from the multiplication table of the group. The orthogonality relation in Theorem 5.1, which is an orthogonality relation for the *rows* of a character table, yield

$$1 + 3\alpha + 2\beta = 0\,, \tag{5.13}$$

$$1 + 3\alpha^2 + 2\beta^2 = 6\,. \tag{5.14}$$

The group multiplication table requires that

$$a^2 = e, \qquad b^2 = e, \qquad c^2 = e, \qquad d^2 = f\,.$$

Since the one-dimensional representations must obey the multiplication table, these products imply that

$$\alpha^2 = 1, \qquad \beta^2 = \beta\,.$$

Substituting these relations into (5.14), yields $4 + 2\beta = 6$, i.e.,

$$\beta = 1$$

Upon substitution of this value into (5.13), we obtain $3 + 3\alpha = 0$, i.e.,

$$\alpha = -1$$

From the orthogonality relation (5.5), which is an orthogonality relation between the *columns* of a character table, we obtain

$$1 + \alpha + 2\gamma = 0$$

$$1 + \beta + 2\delta = 0$$

Substituting the values obtained for α and β into these equations yields

$$\gamma = 0, \qquad \delta = -1$$

The complete character table for S_3 is therefore given by

S_3	$\{e\}$	$\{a,b,c\}$	$\{d,f\}$
Γ_1	1	1	1
Γ_2	1	-1	1
Γ_3	2	0	-1

When character tables are compiled for the most common groups, a notation is used which reflects the fact that the group elements correspond to transformations on physical objects. The notation for the classes of S_3 are as follows:

- $\{e\} \to E$. The identity.

- $\{a, b, c\} \to 3\sigma_v$. Reflection through *vertical* planes, where 'vertical' refers to the fact that these planes contain the axis of highest rotational symmetry, in this case, the z-axis. The '3' refers to there being three elements in this class.

- $\{d, f\} \to 2C_3$. Rotation by $\frac{2}{3}\pi$ radians, with the '2' again referring to the there being two elements in this class. The notation C_3^2 is for rotations by $\frac{4}{3}\pi$ radians, so the 'class' notation is meant only to indicate the *type* of operation. In general, C_n refers to rotations through $2\pi/n$ radians.

Several notations are used for irreducible representations. One of the most common is to use A for one-dimensional representations, E for two-dimensional representations, and T for three-dimensional representations, with subscripts used to distinguish multiple occurrences of irreducible representations of the same dimensionality. The notation Γ is often used to indicate a generic (usually irreducible) representation, with subscripts and superscripts employed to distinguish between different representations. With the first of these conventions, the character table for S_3, which is known as the group C_{3v} when interpreted as the planar symmetry operations of an equilateral triangle, is

C_{3v}	E	$3\sigma_v$	$2C_3$
A_1	1	1	1
A_2	1	-1	1
E	2	0	-1

∎

5.5 Summary

This chapter has been devoted to characters and character tables. The utility of characters in applications stems from the following:

1. The character is a property of the class of an element.

2. Characters are unaffected by similarity transformations, so equivalent representations—reducible or irreducible—have the same characters.

3. As shown in Equations (5.10) and (5.11), the characters of a representation indicate, through a straightforward calculation, whether that representation is reducible or irreducible.

4. Characters of irreducible representations obey orthogonality theorems which, when interpreted in the context of character tables, correspond to the orthogonality relations of their rows and columns.

5. According to the Decomposition Theorem, once the character table of a group is known, the characters of any representation can be decomposed into its irreducible components.