Chapter 7

Continuous Groups, Lie Groups, and Lie Algebras

Zeno was concerned with three problems . . . These are the problem of the infinitesimal, the infinite, and continuity . . .
—Bertrand Russell

The groups we have considered so far have been, in all but a few cases, discrete and finite. Most of the central theorems for these groups and their representations have relied on carrying out sums over the group elements, often in conjunction with the Rearrangement Theorem (Theorem 2.1). These results provide the basis for the application of groups and representations to physical problems through the construction and manipulation of character tables and the associated computations that require direct sums, direct products, orthogonality and decomposition.

But the notion of symmetry transformations that are based on *continuous* quantities also occur naturally in physical applications. For example, the Hamiltonian of a system with spherical symmetry (e.g., atoms and, in particular, the hydrogen atom) is invariant under all three-dimensional rotations. To address the consequences of this invariance within the framework of group theory necessitates confronting several issues that arise from the continuum of rotation angles. These include defining what we mean by a "multiplication table," determining how summations over group elements are carried out, and deriving the appropriate re-statement of the Rearrangement Theorem to enable the Great Orthogonality Theorem and its consequences to be obtained for continuous groups. More generally, the existence of a continuum of group elements, when combined with the requirement of analyticity, introduces new structures associated with constructing differentials and integrals of group elements. In effect, this represents an amalgamation of group theory and analysis, so such groups are the natural objects for describing the symmetry of analytic structures such as differential equations and those that arise in differential geometry. In fact, the introduction of analytic groups by Sophus Lie late in the 19th century was motivated by the search for symmetries of differential equations.

In this chapter we begin our discussion about the modifications to our development of groups and representations that are necessitated by having a continuum of elements. We begin in the first section with the definition of a continuous group and specialize to the most common type of continuous group, the Lie group. We then introduce the idea of an infinitesimal generator of a transformation, from which every element can be obtained by repeated application. These generators embody much of the structure of the group and, because there are a finite number of these entities, are simpler to work with than the full group. This leads naturally to the Lie algebra associated with a Lie group. All of these concepts are illustrated with the groups of proper rotations in two and three dimensions. The representation of these groups, their character tables, and basis functions will be discussed in the next chapter.

7.1 Continuous Groups

Consider a set of elements R that depend on a number of real continuous parameters, $R(a) \equiv R(a_1, a_2, \ldots, a_r)$. These elements are said to form a **continuous group** if they fulfill the requirements of a group (Section 2.1) and if there is some notion of 'proximity' or 'continuity' imposed on the elements of the group in the sense that a small change in one of the factors of a product produces a correspondingly small change in their product. If the group elements depend on r parameters, this is called an r-parameter continuous group.

In general terms, the requirements that a continuous set of elements form a group are the same as those for discrete elements, namely, closure under multiplication, associativity, the existence of a unit, and an inverse for every element. Consider first the multiplication of two elements $R(a)$ and $R(b)$ to yield the product $R(c)$:

$$R(c) = R(a)R(b)\,.$$

Then c must be a continuous real function f of a and b:

$$c = f(a, b)\,.$$

This defines the structure of the group in the same way as the multiplication table does for discrete groups. The associativity of the composition law,

$$R(a)\underbrace{\Big[R(b)R(c)\Big]}_{R[f(b,c)]} = \underbrace{\Big[R(a)R(b)\Big]}_{R[f(b,c)]}R(c)\,,$$

requires that

$$f[a, f(b,c)] = f[f(a,b), c]\,.$$

The existence of an identity element, which we denote by $R(a_0)$,

$$R(a_0)R(a) = R(a)R(a_0) = R(a)\,,$$

is expressed in terms of f as

$$f(a_0, a) = f(a, a_0) = a\,.$$

The inverse of each element $R(a)$, denoted by $R(a')$, produces

$$R(a')R(a) = R(a)R(a') = R(a_0)\,.$$

Therefore,

$$f(a', a) = f(a, a') = a_0\,.$$

If f is an analytic function, i.e., a function with a convergent Taylor series expansion within the domain defined by the parameters, the resulting group is called an r-parameter **Lie group**, named after Sophus Lie, a Norwegian mathematician who provided the foundations for such groups.

Our interest in physical applications centers around transformations on d-dimensional spaces. Examples include Euclidean spaces, where the variables are spatial coordinates, Minkowski spaces, where the variables are space-time coordinates, and spaces associated with internal degrees of freedom, such as spin or isospin. In all cases, these are mappings of the space onto itself and have the general form

$$x_i' = f_i(x_1, x_2, \ldots, x_d; a_1, a_2, \ldots, a_r), \qquad i = 1, 2, \ldots, d\,.$$

If the f_i are analytic, then this defines an r-parameter Lie group of transformations.

Example 7.1 Consider the one-dimensional transformations

$$x' = ax \tag{7.1}$$

where a is an non-zero real number. This transformation corresponds to stretching the real line by a factor a. The product of two such operations, $x'' = ax'$ and $x' = bx$ is

$$x'' = ax' = abx\,.$$

By writing $x'' = cx$, we have that

$$c = ab\,, \tag{7.2}$$

so the multiplication of two transformations is described by an analytic function that yields another transformation of the form in (7.1). This operation is clearly associative, as well as Abelian, since the product transformation corresponds to the multiplication of real numbers. This product can also be used to determine the inverse of these transformations. By setting $c = 1$ in (7.2), so that $x'' = x$, the inverse of (7.1) is seen to correspond to the transformation with $a' = a^{-1}$, which explains the requirement that $a \neq 0$. Finally, the identity is determined from $x' = x$, which clearly corresponds to the transformation with $a = 1$. Hence, the transformations defined in (7.1) form a one-parameter Abelian Lie group. ∎

Example 7.2 Now consider the one-dimensional transformations

$$x' = a_1 x + a_2 \,, \tag{7.3}$$

where again a_1 is an non-zero real number. These transformations corresponds to the stretching of the real line by a factor a_1, as in the preceding Example, and a translation by a_2. The product of two operations is

$$x'' = a_1 x' + a_2 = a_1(b_1 x + b_2) + a_2 = a_1 b_1 x + a_1 b_2 + a_2 \,.$$

By writing $x'' = c_1 x + c_2$, we have that

$$c_1 = a_1 b_1, \qquad c_2 = a_1 b_2 + a_2 \,,$$

so the multiplication of two transformations is described by an analytic function and yields another transformation of the form in (7.1). However, although this multiplication is associative, it is not Abelian, as can be seen from the fact that the indices do not enter symmetrically in c_2. By setting, $c_1 = c_2 = 1$, the inverse of (7.3) is the transformation

$$x' = \frac{x}{a_1} - \frac{a_2}{a_1} \,.$$

The identity is again determined from $x' = x$, which requires that $a_1 = 1$ and $a_2 = 0$. Hence, the transformations in (7.3) form a two-parameter (non-Abelian) Lie group. ∎

7.2 Linear Transformation Groups

An important class of transformations is the group of linear transformations in d dimensions. These can be represented by $d \times d$ matrices. For example, the most general such transformation in two dimensions is $\boldsymbol{x}' = A\boldsymbol{x}$ or, in matrix form,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \,, \tag{7.4}$$

where $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$ (Example 2.4). With no further restriction, and with the composition of two elements given by the usual rules of matrix multiplication, these matrices form a four-parameter Lie group. This Lie group is called the **general linear group** in two dimensions and is denoted by GL(2,R), where the 'R' signifies that the entries are real; the corresponding group with complex entries is denoted by GL(2,C). In n dimensions, these transformation groups are denoted by GL(n,R), or, with complex entries, by GL(n,C).

7.2.1 Orthogonal Groups

Many transformations in physical applications are required to preserve length in the appropriate space. If that space is ordinary Euclidean n-dimensional space, the restriction that lengths be preserved means that

$$x_1'^2 + x_2'^2 + \cdots + x_n'^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \,. \tag{7.5}$$

The corresponding groups, which are subgroups of the general linear group, are called **orthogonal**, and are denoted by O(n).

Consider the orthogonal group in two-dimensions, i.e., O(2), where the coordinates are x and y. By substituting the general transformation (7.4) into (7.5), we require that

$$\begin{aligned} x'^2 + y'^2 &= (a_{11}x + a_{12}y)^2 + (a_{21}x + a_{22}y)^2 \\ &= (a_{11}^2 + a_{21}^2)x^2 + 2(a_{11}a_{12} + a_{21}a_{22})xy + (a_{12}^2 + a_{22}^2)y^2 \,. \end{aligned}$$

For the right-hand side of this equation to be equal to $x^2 + y^2$ for *all* x and y, we must set

$$a_{11}^2 + a_{21}^2 = 1, \qquad a_{11}a_{12} + a_{21}a_{22} = 0, \qquad a_{12}^2 + a_{22}^2 = 1 \,.$$

Thus, we have three conditions imposed on four parameters, leaving one free parameter. These conditions can be used to establish the following relation:

$$(a_{11}a_{22} - a_{12}a_{21})^2 = 1 \,.$$

Recognizing the quantity in parentheses as the determinant of the transformation, this condition implies that

$$\det(A) = \pm 1\,.$$

If $\det(A) = 1$, then the parity of the coordinate system is not changed by the transformation; this corresponds to a *proper* rotation. If $\det(A) = -1$, then the parity of the coordinate system is changed by the transformation; this corresponds to an *improper* rotation. As we have already seen, both types of transformations are important in physical applications, but we will first examine the proper rotations in two-dimensions. This group is called the *special* orthogonal group in two dimensions and is denoted by SO(2), where "special" signifies the restriction to proper rotations. The parametrization of this group that we will use is

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \tag{7.6}$$

where φ, the single parameter in this Lie group, is the rotation angle of the transformation. As can easily be checked using the trigonometric identities for the sum of two angles,

$$R(\varphi_1 + \varphi_2) = R(\varphi_1)R(\varphi_2)\,, \tag{7.7}$$

so this group is clearly Abelian.

7.3 Infinitesimal Generators

A construction of immense utility in the study of Lie groups, which was introduced and extensively studied by Lie, is the **infinitesimal generator**. The idea behind this is that instead of having to consider the group as a whole, for many purposes it is sufficient to consider an infinitesimal transformation around the identity. Any finite transformation can then be constructed by the repeated application, or "integration," of this infinitesimal transformation.

7.3.1 Matrix Form of Generators

For SO(2), we first expand $R(\varphi)$ in a Taylor series around the identity ($\varphi = 0$):

$$R(\varphi) = R(0) + \frac{\mathrm{d}R}{\mathrm{d}\varphi}\bigg|_{\varphi=0}\varphi + \frac{1}{2}\frac{\mathrm{d}^2R}{\mathrm{d}\varphi^2}\bigg|_{\varphi=0}\varphi^2 + \cdots . \tag{7.8}$$

The coefficients in this series can be determined directly from (7.6), but a more elegant solution may be found by first differentiating (7.7) with respect to φ_1,

$$\frac{\mathrm{d}}{\mathrm{d}\varphi_1}R(\varphi_1 + \varphi_2) = \frac{\mathrm{d}R(\varphi_1)}{\mathrm{d}\varphi_1}R(\varphi_2)\,, \tag{7.9}$$

then setting $\varphi_1 = 0$. Using the chain rule, the left-hand side of this equation is

$$\left[\frac{\mathrm{d}R(\varphi_1 + \varphi_2)}{\mathrm{d}(\varphi_1 + \varphi_2)}\,\frac{\mathrm{d}(\varphi_1 + \varphi_2)}{\mathrm{d}\varphi_1}\right]\bigg|_{\varphi_1=0} = \frac{\mathrm{d}R(\varphi_2)}{\mathrm{d}\varphi_2}\,,$$

so Eq. (7.9) becomes

$$\frac{\mathrm{d}R(\varphi)}{\mathrm{d}\varphi} = XR(\varphi)\,, \tag{7.10}$$

where

$$\frac{\mathrm{d}R(\varphi_1)}{\mathrm{d}\varphi_1}\bigg|_{\varphi_1=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \equiv X\,. \tag{7.11}$$

Equations (7.10) and (7.11) allow us to determine all of the expansions coefficients in (7.9). By setting $\varphi = 0$ in (7.10) and observing that $R(0) = I$, where I is the 2×2 unit matrix,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,,$$

we obtain

$$\frac{\mathrm{d}R(\varphi)}{\mathrm{d}\varphi}\bigg|_{\varphi=0} = X\,. \tag{7.12}$$

To determine the higher-order derivatives of R, we differentiate (7.10) n times, and set $\varphi = 0$:

$$\frac{\mathrm{d}^n R(\varphi)}{\mathrm{d}\varphi^n}\bigg|_{\varphi=0} = X\frac{\mathrm{d}^{n-1} R(\varphi)}{\mathrm{d}\varphi^{n-1}}\bigg|_{\varphi=0}.$$

This yields, in conjunction with (7.12),

$$\frac{\mathrm{d}^n R(\varphi)}{\mathrm{d}\varphi^n}\bigg|_{\varphi=0} = X^n.$$

Substituting this expression into the Taylor series in (7.8) allows us to write

$$\begin{aligned} R(\varphi) &= I + X\varphi + \tfrac{1}{2}X^2\varphi^2 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(X\varphi)^n \\ &\equiv \mathrm{e}^{\varphi X}, \end{aligned}$$

where $X^0 = I$ and the exponential of a matrix is *defined* by the Taylor series expansion of the exponential. Thus, every rotation by a finite angle can be obtain from the exponentiation of the matrix X, which is called the **infinitesimal generator** of rotations. Since $X^2 = I$, it is a straightforward matter to show directly from the Taylor series of the exponential (Problem 4, Problem Set 9) that

$$\mathrm{e}^{\varphi X} = I\cos\varphi + X\sin\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}.$$

7.3.2 Operator Form of Generators

An alternative way of representing infinitesimal generators through which connections with quantum mechanics can be directly made is in terms of differential operators. To derive the operator associated with infinitesimal rotations, we expand (7.6) to first order in $\mathrm{d}\varphi$ to obtain the transformation

$$\begin{aligned} x' &= x\cos\varphi - y\sin\varphi = x - y\,\mathrm{d}\varphi, \\ y' &= x\sin\varphi + y\cos\varphi = x\,\mathrm{d}\varphi + y. \end{aligned}$$

An arbitrary differentiable function $F(x, y)$ then transforms as

$$F(x', y') = F(x - y\,\mathrm{d}\varphi, x\,\mathrm{d}\varphi + y)\,.$$

Retaining terms to first order in $\mathrm{d}\varphi$ on the right-hand side of this equation yields

$$F(x', y') = F(x, y) + \Big(-y\frac{\partial F}{\partial x} + x\frac{\partial F}{\partial y}\Big)\,\mathrm{d}\varphi\,.$$

Since F is an arbitrary function, we can associate infinitesimal rotations with the operator

$$X = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\,.$$

As we will see in the next section, this operator is proportional to the z-component of the angular momentum operator.

The group SO(2) is simple enough that the full benefits of an infinitesimal generator are not readily apparent. We will see in the next section, where we discuss SO(3), that the infinitesimal generators embody much of the structure of the full group.

7.4 SO(3)

The orthogonal group in three dimensions is comprised of the transformations that leave the quantity $x^2 + y^2 + z^2$ invariant. The group GL(3,R) has 9 parameters, but the invariance of the length produces six independent conditions, leaving three free parameters, so O(3) forms a three-parameter Lie group. If we restrict ourselves to transformations with unit determinant, we obtain the group of proper rotations in three dimensions, SO(3).

There are three common ways to parametrize these rotations:

- Successive rotations about three mutually orthogonal *fixed* axes.

- Successive about the z-axis, about the *new* y-axis, and then about the *new* z-axis. These are called **Euler angles**.

- The axis-angle representation, defined in terms of an axis whose direction is specified by a unit vector (two parameters) and a rotation about that axis (one parameter).

In this section, we will use the first of these parametrizations to demonstrate some of the properties of SO(3). In the next chapter, where we will develop the orthogonality relations for this group, the axis-angle representation will prove more convenient.

7.4.1 Rotation Matrices

Consider first rotations about the z-axis by an angle φ_3:

$$R_3(\varphi_3) = \begin{pmatrix} \cos\varphi_3 & -\sin\varphi_3 & 0 \\ \sin\varphi_3 & \cos\varphi_3 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

The corresponding infinitesimal generator is calculated as in (7.11):

$$X_3 = \frac{\mathrm{d}R_3}{\mathrm{d}\varphi_3}\bigg|_{\varphi_3=0} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} .$$

These results are essentially identical to those for SO(2). However, for SO(3), we have rotations about two other axes to consider. For rotations about the x-axes by an angle φ_1, the rotation matrix is

$$R_1(\varphi_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & -\sin\varphi_1 \\ 0 & \sin\varphi_1 & \cos\varphi_1 \end{pmatrix}$$

and the corresponding generator is

$$X_1 = \frac{\mathrm{d}R_1}{\mathrm{d}\varphi_1}\bigg|_{\varphi_1=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Finally, for rotations about the y-axis by an angle φ_2, we have

$$R_2(\varphi_2) = \begin{pmatrix} \cos\varphi_2 & 0 & \sin\varphi_2 \\ 0 & 1 & 0 \\ -\sin\varphi_2 & 0 & \cos\varphi_2 \end{pmatrix}$$

and the generator is

$$X_2 = \frac{\mathrm{d}R_2}{\mathrm{d}\varphi_2}\bigg|_{\varphi_2=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

As can be easily verified, the matrices $R_i(\varphi_i)$ do not commute, nor do the X_i. However, the X_i have an additional useful property, namely closure under commutation. As an example, consider the products X_1X_2 and X_2X_1:

$$X_1X_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$X_2X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus, the commutator of X_1 and X_2, denoted by $[X_1, X_2]$ is given by

$$[X_1, X_2] \equiv X_1X_2 - X_2X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = X_3$$

Similarly, we have

$$[X_2, X_3] = X_1, \qquad [X_3, X_1] = X_2$$

The commutation relations among all of the X_i can be succinctly summarized by introducing the anti-symmetric symbol ε_{ijk}, which takes the value 1 for a symmetric permutation of distinct i, j, and k, the value -1 for an antisymmetric permutation, and is zero otherwise (i.e., if two or more of i, j and k are equal). We can then write

$$[X_i, X_j] = \varepsilon_{ijk} X_k \tag{7.13}$$

We will discuss the physical interpretation of these generators once we obtain their operator form in the next section.

7.4.2 Operators for Infinitesimal Rotations

As was the case in Section 7.3, an alternative to the matrix representation of infinitesimal generators is in terms of differential operators. Proceeding as in that section, we first write the general rotation as an expansion to first order in each of the φ_i about the identity. This yields the transformation matrix

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & -\varphi_3 & \varphi_2 \\ \varphi_3 & 1 & -\varphi_1 \\ -\varphi_2 & \varphi_1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Substituting this coordinate transformation into a differentiable function $F(x, y, z)$,

$$F(x', y', z') = F(x - \varphi_3 y + \varphi_2 z, y + \varphi_3 x - \varphi_1 z, z - \varphi_2 x + \varphi_1 y)$$

and expanding the right-hand side to first order in the φ_i yields the following expression:

$$\begin{aligned} F(x', y', z') &= F(x, y, z) \\ &+ \Big(\frac{\partial F}{\partial z} y - \frac{\partial F}{\partial y} z\Big)\varphi_1 + \Big(\frac{\partial F}{\partial x} z - \frac{\partial F}{\partial z} x\Big)\varphi_2 + \Big(\frac{\partial F}{\partial y} x - \frac{\partial F}{\partial x} y\Big)\varphi_3 \end{aligned}$$

Since F is an arbitrary differentiable function, we can identify the generators X_i of rotations about the coordinate axes from the coefficients of the φ_i, i.e., with the differential operators

$$X_1 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

$$X_2 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \tag{7.14}$$

$$X_3 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

Notice that X_3 is the operator obtained for SO(2) in Section 7.3. We can now assign a physical interpretation to these operators by comparing them with the vectors components of the angular operators in the coordinate representation, obtained from the definition

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times (-i\hbar\boldsymbol{\nabla})$$

Carrying out the cross-product yields the standard expressions

$$L_1 = -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_2 = -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{7.15}$$

$$L_3 = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

for the x, y, and z components of $\boldsymbol{L}$, respectively. Thus, $L_i = -i\hbar X_i$, for $i = 1, 2, 3$, and (7.13) becomes

$$[L_i, L_j] = i\hbar\varepsilon_{ijk}L_k$$

which are the usual angular momentum commutation relations. Therefore, we can associate the vector components of the angular momentum operator with the generators of infinitesimal rotations about the corresponding axes. An analogous association exists between the vector components of the coordinate representation of the linear momentum operator and differential translation operations along the corresponding directions.

7.4.3 The Algebra of Infinitesimal Generators

The commutation relations in (7.13) define a "product" of two generators which yields the third generator. Thus, the set of generators is closed under this operation. Triple products, which determine whether or not this composition law is associative, can be written in a concise form using only the definition of the commutator, i.e., in the form of an identity, without any explicit reference to the quantities involved. Beginning with the triple product

$$\begin{aligned}\big[A,[B,C]\big] &= A[B,C]-[B,C]A \\ &= ABC-ACB-BCA+CBA\end{aligned}$$

We now add and subtract the quantities BAC and CAB on the right-hand side of this equation and rearrange the resulting expression into commutators to obtain

$$\begin{aligned}\big[A,[B,C]\big] &= ABC-ACB-BCA+CBA \\ &\quad +BAC-BAC+CAB-CAB \\ &= -C(AB-BA)+(AB-BA)C \\ &\quad +B(AC-CA)-(AC-CA)B \\ &= -\big[[A,B],C\big]+\big[[C,A],B\big]\end{aligned}$$

A simple rearrangement yields the **Jacobi identity**:

$$\big[A,[B,C]\big]+\big[B,[C,A]\big]+\big[C,[A,B]\big]=0$$

Notice that this identity has been obtained using only the definition of the commutator.

For the infinitesimal generators of the rotation group, with the commutator in (7.13), each of the terms in the Jacobi identity vanishes. Thus,

$$\big[A,[B,C]\big]=\big[[A,B],C\big]$$

so the product of these generators is associative. In the more general case, however, products of quantities defined in terms of a commutator are not associative. The **Lie algebra** associated with the Lie group from which the generators are obtained consists of quantities $A, B, C, \ldots$ defined by

$$A = \sum_{k=1}^{3} a_k X_k, \qquad B = \sum_{k=1}^{3} b_k X_k, \qquad C = \sum_{k=1}^{3} c_k X_k, \qquad \text{etc.}$$

where the $a_k, b_k, c_k, \ldots$ are real coefficients and from which linear combinations $\alpha A + \beta B$ with real α and β can be formed. The product is given by

$$[A, B] = -[B, A]$$

and the Jacobi identity is, of course, satisfied.

The formal definition of a Lie algebra, which is an abstraction of the properties just discussed, is as follows.

Definition. A **Lie algebra** is a vector space L over some field F[1] (typically the real or complex numbers) together with a binary operation $[\cdot, \cdot] : L \times L \to L$, called the *Lie bracket*, which has the following properties:

1. **Bilinearity**.

$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[z, ax + by] = a[z, x] + b[z, y]$$

   for all a and b in F and x, y, and z in L.

2. **Jacobi identity**.

$$\big[[x, y], z\big] + \big[[z, x], y\big] + \big[[y, z], x\big] = 0$$

   for all x, y, and z in L.

[1] A field is an algebraic system of elements in which the operations of addition, subtraction, multiplication, and division (except by zero) may be performed without leaving the system (closure) and the associative, commutative, and distributive rules, familiar from the arithmetic of ordinary numbers, hold. Examples of fields are the rational numbers, the real numbers, and the complex numbers. The smallest field has only two elements: $\{0, 1\}$. The concept of a field is useful for defining vectors and matrices, whose components can be elements of any field.

3. **Antisymmetry**.

$$[x, y] = -[y, x]$$

for all x and y in L.

7.5 Summary

In this chapter, we have described the properties of Lie groups in terms of specific examples, especially SO(2) and SO(3). With this background, we can generalize our discussion to any Lie group. An r-parameter Lie group of transformations on an n-dimensional space is

$$x_i' = f_i(x_1, x_2, \ldots, x_n; a_1, a_2, \ldots, a_r)$$

where $i = 1, 2, \ldots, n$. If only one of the r parameters a_i is changed from zero, while all the other parameters are held fixed, we obtain the infinitesimal transformations X_i associated with this Lie group. These can be expressed as differential operators by examining the effect of these infinitesimal coordinate transformations on an arbitrary differentiable function F:

$$\begin{aligned} \mathrm{d}F &= \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}\,\mathrm{d}x_j \\ &= \sum_{j=1}^{n} \frac{\partial F}{\partial x_j}\biggl(\sum_{i=1}^{r} \frac{\partial f_j}{\partial a_i}\bigg|_{a=0} \mathrm{d}a_i\biggr) \\ &= \sum_{i=1}^{r} \mathrm{d}a_i \biggl(\sum_{j=1}^{n} \frac{\partial f_j}{\partial a_i}\bigg|_{a=0} \frac{\partial}{\partial x_j}\biggr) F \end{aligned}$$

We identify the differential operators X_i as the coefficient of $\mathrm{d}a_i$ in this differential:

$$X_i = \sum_{j=1}^{n} \frac{\partial f_j}{\partial a_i}\bigg|_{a=0} \frac{\partial}{\partial x_j}$$

for $r = 1, 2, \ldots, r$. These operators satisfy commutation relations of the form

$$[X_i, X_j] = c_{ij}^k X_k$$

where the c_{ij}^k are called **structure constants** and are a property of the group. The commutator satisfies the Jacobi identity,

$$\Big[X_i, [X_j, X_k]\Big] + \Big[X_j, [X_k, X_i]\Big] + \Big[X_k, [X_i, X_j]\Big] = 0$$

which places a constraint on the structure constants. The commutator and the Jacobi identity, together with the ability to form real linear combinations of the X_i endows these generators with the structure of an algebra, called the Lie algebra associated with the Lie group.