Group Theory

Problem Set 1 October 12, 2001

Note: Problems marked with an asterisk are for Rapid Feedback; problems marked with a double asterisk are optional.

1. Show that the wave equation for the propagation of an impulse at the speed of light c,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\,,$$

is covariant under the Lorentz transformation

$$x'=\gamma(x-vt),\quad y'=y,\quad z'=z,\quad t'=\gamma\left(t-\frac{v}{c^2}x\right),$$

where $\gamma=(1-v^2/c^2)^{-1/2}$.

2.* The Schrödinger equation for a free particle of mass m is

$$i\hbar\frac{\partial\varphi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x^2}\,.$$

Show that this equation is invariant to the global change of phase of the wavefunction:

$$\varphi\to\varphi'=\mathrm{e}^{i\alpha}\varphi\,,$$

where α is any real number. This is an example of an **internal** symmetry transformation, since it does not involve the space-time coordinates.

According to Noether's theorem, this symmetry implies the existence of a conservation law. Show that the quantity $\int_{-\infty}^{\infty}|\varphi(x,t)|^2\,dx$ is independent of time for solutions of the free-particle Schrödinger equation.

3.* Consider the following sets of elements and composition laws. Determine whether they are groups and, if not, identify which group property is violated.

(a) The rational numbers, excluding zero, under multiplication.

(b) The non-negative integers under addition.

(c) The even integers under addition.

(d) The nth roots of unity, i.e., $\mathrm{e}^{2\pi mi/n}$, for $m=0,1,\ldots,n-1$, under multiplication.

(e) The set of integers under ordinary *subtraction*.

4.** The general form of the Liouville equation is

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \big[q(x) + \lambda r(x)\big]y = 0$$

where p, q and r are real-valued functions of x with p and r taking only positive values. The quantity λ is called the eigenvalue and the function y, called the eigenfunction, is assumed to be defined over an interval $[a, b]$. We take the boundary conditions to be

$$y(a) = y(b) = 0$$

but the result derived below is also valid for more general boundary conditions. Notice that the Liouville equations contains the one-dimensional Schrödinger equation as a special case.

Let $u(x;\lambda)$ and $v(x;\lambda)$ be the fundamental solutions of the Liouville equation, i.e. u and v are two linearly-independent solutions in terms of which all other solutions may be expressed (for a given value λ). Then there are constants A and B which allow any solution y to be expressed as a linear combination of this fundamental set:

$$y(x;\lambda) = Au(x;\lambda) + Bv(x;\lambda)$$

These constants are determined by requiring $y(x;\lambda)$ to satisfy the boundary conditions:

$$y(a;\lambda) = Au(a;\lambda) + Bv(a;\lambda) = 0$$

$$y(b;\lambda) = Au(b;\lambda) + Bv(b;\lambda) = 0$$

Use this to show that the solution $y(x;\lambda)$ is unique, i.e., that there is one and only one solution corresponding to an eigenvalue of the Liouville equation.