# Group Theory

Problem Set 10 December 11, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

**1.*** Prove that a proper orthogonal transformation in an odd-dimensional space always possesses an axis, i.e., a line whose point are left unchanged by the transformation.

**2.** Prove **Euler's theorem**: The general displacement of a rigid body with one fixed point is a rotation about an axis.

**3.*** The functions $(x \pm iy)^m$, where $m$ is an integer generate irreducible representations of SO(2). Suppose we now consider the group O(2), where we now allow *improper* rotations. Use Schur's lemma to show that these functions generate irreducible *two*-dimensional representations of O(2) for $m \neq 0$, but a *reducible* representation for $m = 0$.

Hint: The general improper rotation in two dimensions is

$$\begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}$$

**4.** Consider the rotation matrix obtained by rotating an initial set of axes counterclockwise by $\phi$ about the $z$-axis, then rotated about the new $x$-axis counterclockwise by $\theta$, and finally rotated about the new $z$-axis counterclockwise by $\psi$. These are the **Euler angles** and the corresponding rotation matrix is

$$\begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\theta\sin\phi\cos\psi & -\sin\psi\sin\phi + \cos\theta\cos\phi\cos\psi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Verify that the angle of rotation $\varphi$ of this transformation is given by

$$\cos\left(\tfrac{1}{2}\varphi\right) = \cos\left[\tfrac{1}{2}(\phi + \psi)\right]\cos\left(\tfrac{1}{2}\theta\right)$$

**5.** Determine the axis of the transformation in Problem 4.

**6.*** Verify that the direct product of two irreducible representations of SO(3) has the following decomposition

$$\chi^{(\ell_1)}(\varphi)\chi^{(\ell_2)}(\varphi) = \sum_{\ell=|\ell_1-\ell_2|}^{\ell_1+\ell_2} \chi^{(\ell)}(\varphi)$$

This is called the **Clebsch–Gordan series** and provides a group-theoretic statement of the addition of angular momenta.

**7.*** Determine the corresponding Clebsch-Gordan series for SO(2).

**8.*** Show that the requirement that $xx^* + yy^*$ is invariant under the complex transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

together with the determinant of this transformation being unity means that the transformation must have the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $aa^* + bb^* = 1$.