Group Theory

Solutions to Problem Set 10 December 14, 2001

1. As shown in Section 8.3.1, the eigenvalues of an orthogonal matrix have modulus unity. These eigenvalues are also the roots of the polynomial equation $\det(A - \lambda I) = 0$, so the Fundamental Theorem of Algebra requires that, if these roots are complex, they must occur in complex conjugate pairs. Thus, only in an *odd-dimensional* space is there guaranteed to be a single real eigenvalue of unity. The corresponding eigenvector is the axis of rotation.

2. If the fixed point is taken as the origin of the set of axes of the body, then the displacement of the rigid body involves no translation, but only a change of orientation, i.e., a rotation. Since, in three dimensions, every rotation can be expressed in an axis-angle representation, Euler's theorem follows immediately.

3. The general improper transformation in two dimensions is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

Thus, for the functions $(x \pm iy)^m$ we have

$$\begin{aligned} (x' \pm iy')^m &= \Big[x\cos\varphi + y\sin\varphi \pm i(x\sin\varphi - y\cos\varphi)\Big]^m \\ &= \Big[x(\cos\varphi \pm i\sin\varphi) \mp iy(\cos\varphi \pm i\sin\varphi)\Big]^m \\ &= (x \mp iy)^m\, \mathrm{e}^{\pm im\varphi} , \end{aligned}$$

so they generate the representation

$$\begin{bmatrix} (x' + iy')^m \\ (x' - iy')^m \end{bmatrix} = \begin{pmatrix} 0 & \mathrm{e}^{im\varphi} \\ \mathrm{e}^{-im\varphi} & 0 \end{pmatrix} \begin{bmatrix} (x + iy)^m \\ (x - iy)^m \end{bmatrix} .$$

To determine whether or not this representation is reducible, we apply Schur's first lemma. Suppose a matrix A commutes with all of the matrices of our two dimensional representation. Then, we have

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 & \mathrm{e}^{im\varphi} \\ \mathrm{e}^{-im\varphi} & 0 \end{pmatrix}}_{\begin{pmatrix} a_{12}\mathrm{e}^{-im\varphi} & a_{11}\mathrm{e}^{im\varphi} \\ a_{22}\mathrm{e}^{-im\varphi} & a_{21}\mathrm{e}^{im\varphi} \end{pmatrix}} = \underbrace{\begin{pmatrix} 0 & \mathrm{e}^{im\varphi} \\ \mathrm{e}^{-im\varphi} & 0 \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{\begin{pmatrix} a_{21}\mathrm{e}^{im\varphi} & a_{22}\mathrm{e}^{im\varphi} \\ a_{11}\mathrm{e}^{-im\varphi} & a_{12}\mathrm{e}^{-im\varphi} \end{pmatrix}} .$$

Thus, if $m \neq 0$, we must require that $a_{12} = a_{21} = 0$ and that $a_{11} = a_{22}$, i.e., A is multiple of the 2×2 unit matrix and, according to Schur's first lemma, this representation is *irreducible*. However, of $m = 0$, we need only require that $a_{12} = a_{21}$ and $a_{11} = a_{22}$, so this is a *reducible* representation.

4. The rotation angle φ is calculated from the trace of the transformation matrix:

$$\begin{aligned} 1 + 2\cos\varphi &= \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi - \sin\psi\sin\phi \\ &\qquad + \cos\theta\cos\phi\cos\psi + \cos\theta \\ &= (1+\cos\theta)(\cos\phi\cos\psi - \sin\phi\sin\psi) + \cos\theta \\ &= (1+\cos\theta)\cos(\phi+\psi) + \cos\theta\,. \end{aligned}$$

Using the triginometric identity

$$1 + 2\cos\varphi = 4\cos^2\left(\tfrac{1}{2}\varphi\right) - 1\,,$$

we obtain

$$\begin{aligned} 4\cos^2\left(\tfrac{1}{2}\varphi\right) &= (1+\cos\theta)[1+\cos(\phi+\psi)] \\ &= 4\cos^2\left(\tfrac{1}{2}\theta\right)\cos^2\left[\tfrac{1}{2}(\phi+\psi)\right], \end{aligned}$$

or,

$$\cos\left(\tfrac{1}{2}\varphi\right) = \cos\left(\tfrac{1}{2}\theta\right)\cos\left[\tfrac{1}{2}(\phi+\psi)\right].$$

5. The axis of the transformation in Problem 4 is determined from the equations derived in Section 8.3.2:

$$\frac{n_2}{n_1} = \frac{a_{31}-a_{13}}{a_{23}-a_{32}}, \qquad \frac{n_3}{n_1} = \frac{a_{12}-a_{21}}{a_{23}-a_{32}}.$$

The denominator of these expressions is

$$a_{23}-a_{32} = \sin\theta\cos\psi + \sin\theta\cos\phi = \sin\theta(\cos\psi+\cos\phi).$$

We also have

$$\begin{aligned} a_{31}-a_{13} &= \sin\theta\sin\phi - \sin\theta\sin\psi = \sin\theta(\sin\phi-\sin\psi) \\ a_{12}-a_{21} &= \cos\psi\sin\phi + \cos\theta\cos\phi\sin\psi \\ &\qquad + \sin\psi\cos\phi + \cos\theta\sin\phi\cos\psi \\ &= (1+\cos\theta)(\cos\phi\sin\psi + \sin\phi\cos\psi) \\ &= (1+\cos\theta)\sin(\phi+\psi). \end{aligned}$$

Thus, the (unnormalized) direction of the rotation axis is

$$\left\{1,\ \frac{\sin\phi-\sin\psi}{\cos\psi+\cos\phi},\ 2\frac{(1+\cos\theta)\sin(\phi+\psi)}{\sin\theta(\cos\phi+\cos\psi)}\right\}.$$

6. There are a number of ways of decomposing the direct product of irreducible representations of SO(3). The books by Tinkham

and Jones give two very different approaches. Below, we provide a third method. We first calculate the direct product

$$\chi^{(\ell)}(\varphi)\chi^{(1)}(\varphi) = \left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{-im\varphi}\right)\left(\sum_{m_1=-1}^{1} \mathrm{e}^{-im_1\varphi}\right).$$

By expanding the second summation and multiplying the first summation with each of the exponentials, we obtain

$$\begin{aligned}\left(\sum_{m=-\ell}^{\ell} \mathrm{e}^{-im\varphi}\right)\left(\sum_{m_1=-1}^{1} \mathrm{e}^{-im_1\varphi}\right) &= \sum_{m=-\ell}^{\ell} \mathrm{e}^{-im\varphi}\left(\mathrm{e}^{i\varphi} + 1 + \mathrm{e}^{-i\varphi}\right) \\ &= \sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m-1)\varphi} + \sum_{m=-\ell}^{\ell} \mathrm{e}^{-im\varphi} + \sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m+1)\varphi}.\end{aligned}$$

If, in the first summation on the right-hand side of this equation, we change the summation variable to $m' = m - 1$ and in the last summation change the summation variable to $m' = m + 1$, we have

$$\begin{aligned}&\sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m-1)\varphi} + \sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m+1)\varphi} \\ &\qquad = \sum_{m'=-\ell-1}^{\ell-1} \mathrm{e}^{-im'\varphi} + \sum_{m'=-\ell+1}^{\ell+1} \mathrm{e}^{-im'\varphi} \\ &\qquad = \sum_{m'=-(\ell+1)}^{\ell+1} \mathrm{e}^{-im'\varphi} + \sum_{m'=-(\ell-1)}^{\ell-1} \mathrm{e}^{-im'\varphi}.\end{aligned}$$

In fact, for any positive integer k, we have

$$\begin{aligned}&\sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m-k)\varphi} + \sum_{m=-\ell}^{\ell} \mathrm{e}^{-i(m+k)\varphi} \\ &\qquad = \sum_{m'=-\ell-k}^{\ell-k} \mathrm{e}^{-im'\varphi} + \sum_{m'=-\ell+k}^{\ell+k} \mathrm{e}^{-im'\varphi} \\ &\qquad = \sum_{m'=-(\ell+k)}^{\ell+k} \mathrm{e}^{-im'\varphi} + \sum_{m'=-(\ell-k)}^{\ell-k} \mathrm{e}^{-im'\varphi}. \qquad (1)\end{aligned}$$

Thus, we conclude that

$$\chi^{(\ell)}(\varphi)\chi^{(1)}(\varphi) = \chi^{(\ell-1)}(\varphi) + \chi^{(\ell)}(\varphi) + \chi^{(\ell+1)}(\varphi)\,.$$

Then, by using (1), we have, in the general case

$$\begin{aligned}
\chi^{(\ell)}(\varphi)\chi^{(\ell')}(\varphi) &= \sum_{m=-\ell}^{\ell} \mathrm{e}^{-im\varphi}\Big[\mathrm{e}^{i\ell'\varphi} + \mathrm{e}^{i(\ell'-1)\varphi} + \cdots + \mathrm{e}^{-i\ell'\varphi}\Big] \\
&= \chi^{(\ell+\ell')}(\varphi) + \chi^{(\ell+\ell'-1)}(\varphi) + \cdots + \chi^{(\ell-\ell')}\,.
\end{aligned}$$

Therefore,

$$\chi^{(\ell)}(\varphi)\chi^{\ell'}(\varphi) = \sum_{m=\ell-\ell'}^{\ell+\ell'} \chi^{(m)}(\varphi)\,,$$

where, from our procedure, it is clear that $\ell' \le \ell$.

7. The corresponding Clebsch–Gordan series for SO(2) is very simple because the group is Abelian. Since

$$\chi^{(m)}(\varphi) = \mathrm{e}^{im\varphi}\,,$$

then

$$\chi^{(m_1)}(\varphi)\chi^{(m_2)}(\varphi) = \chi^{(m_1+m_2)}(\varphi)\,.$$

8. Given the complex transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{2}$$

then the invariance of the quantity $xx^* + yy^*$ yields

$$\begin{aligned}
&x'x'^* + y'y'^* \\
&= (ax + by)(a^*x^* + b^*y^*) + (cx + dy)(c^*x^* + d^*y^*) \\
&= (aa^* + cc^*)xx^* + (ab^* + cd^*)xy^* + (a^*b + c^*d)x^*y \\
&+ (cc^* + dd^*)yy^* \,.
\end{aligned}$$

Maintaining equality for all independent variations of x and y requires that

$$aa^* + cc^* = 1, \qquad ab^* + cd^* = 0, \qquad cc^* + dd^* = 1\,. \tag{3}$$

A fourth condition is that the determinant of the transformation in (2) is unity:

$$ad - bc = 1 \tag{4}$$

If we take the second of equations (3), multiply by a^*, and then use the first of these equations and Equation (4), we obtain

$$\begin{aligned}
a^*(ab^* + cd^*) &= aa^*b^* + a^*cd^* \\
&= (1 - cc^*)b^* + (1 + b^*c^*)c \\
&= b^* + c = 0
\end{aligned}$$

which yields

$$c = -b^*$$

The second of equations (4) then immediately yields

$$a = d^* \tag{5}$$

Thus, the transformation (2) must have the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $aa^* + bb^* = 1$.