Group Theory

Solutions to Problem Set 1 October 6, 2000

1. To express the the wave equation for the propagation of an impulse at the speed of light c,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\,, \tag{1}$$

under the Lorentz transformation

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\Big(t - \frac{v}{c^2}x\Big)\,, \tag{2}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$, we need to obtain expressions for the second derivatives in the primed variables. With $u'(x', y', z') = u(x, y, t)$, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u'}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u'}{\partial t'}\frac{\partial t'}{\partial x} = \gamma\frac{\partial u'}{\partial x'} - \gamma\frac{v}{c^2}\frac{\partial u'}{\partial t'}\,, \tag{3}$$

$$\frac{\partial^2 u}{\partial x^2} = \gamma^2\frac{\partial^2 u'}{\partial x'^2} + \gamma^2\frac{v^2}{c^4}\frac{\partial^2 u'}{\partial t'^2}\,, \tag{4}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u'}{\partial y'^2}\,, \tag{5}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u'}{\partial z'^2}\,, \tag{6}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u'}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial u'}{\partial x'}\frac{\partial x'}{\partial t} = \gamma\frac{\partial u'}{\partial t'} - \gamma v\frac{\partial u'}{\partial x'}\,, \tag{7}$$

$$\frac{\partial^2 u}{\partial t^2} = \gamma^2\frac{\partial^2 u'}{\partial t'^2} + \gamma^2 v^2\frac{\partial^2 u'}{\partial x'^2}\,. \tag{8}$$

Substituting these expressions into the wave equation yields

$$\frac{\gamma^2}{c^2}\frac{\partial^2 u'}{\partial t'^2} + \frac{\gamma^2 v^2}{c^2}\frac{\partial^2 u'}{\partial x'^2} = \gamma^2\frac{\partial^2 u'}{\partial x'^2} + \gamma^2\frac{v^2}{c^4}\frac{\partial^2 u'}{\partial t'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\,, \tag{9}$$

which, upon rearrangement, becomes

$$\frac{\gamma^2}{c^2}\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2 u'}{\partial t'^2}=\gamma^2\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}+\frac{\partial^2 u'}{\partial z'^2}\,. \tag{10}$$

Invoking the definition of γ, we obtain

$$\frac{1}{c^2}\frac{\partial^2 u'}{\partial t'^2}=\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}+\frac{\partial^2 u'}{\partial z'^2}\,, \tag{11}$$

which confirms the covariance of the wave equation under the Lorentz transformation.

2. We begin with the Schrödinger equation for a free particle of mass m:

$$i\hbar\frac{\partial\varphi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\varphi}{\partial x^2}\,. \tag{12}$$

Performing the transformation

$$\varphi\to\varphi'=\mathrm{e}^{i\alpha}\varphi\,, \tag{13}$$

where α is any real number, we find

$$i\hbar\mathrm{e}^{i\alpha}\frac{\partial\varphi'}{\partial t}=-\frac{\hbar^2}{2m}\mathrm{e}^{i\alpha}\frac{\partial^2\varphi'}{\partial x^2}\,, \tag{14}$$

or, upon cancelling the common factor $\mathrm{e}^{i\alpha}$,

$$i\hbar\frac{\partial\varphi'}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\varphi'}{\partial x^2}\,, \tag{15}$$

which establishes the covariance of the Schrödinger equation under this transformation.

We can derive the corresponding quantity multiplying Eq. (12) by the complex conjugate φ^* and subtracting the product of φ and the complex conjugate of Eq. (12):

$$i\hbar\left(\varphi^*\frac{\partial\varphi}{\partial t}+\varphi\frac{\partial\varphi^*}{\partial t}\right)=-\frac{\hbar^2}{2m}\left(\varphi^*\frac{\partial^2\varphi}{\partial x^2}-\varphi\frac{\partial^2\varphi^*}{\partial x^2}\right). \tag{16}$$

The left-hand side of this equation can be written as

$$i\hbar\left(\varphi^*\frac{\partial\varphi}{\partial t}+\varphi\frac{\partial\varphi^*}{\partial t}\right)=i\hbar\frac{\partial}{\partial t}\left(\varphi\varphi^*\right)=i\hbar\frac{\partial}{\partial t}|\varphi|^2\,, \tag{17}$$

and the right-hand side can be written as

$$-\frac{\hbar^2}{2m}\left(\varphi^*\frac{\partial^2\varphi}{\partial x^2}-\varphi\frac{\partial^2\varphi^*}{\partial x^2}\right)=-\frac{\hbar^2}{2m}\frac{\partial}{\partial x}\left(\varphi^*\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial\varphi^*}{\partial x}\right). \tag{18}$$

We now consider the solution to Eq. (12) corresponding to a **wave packet**, whereby by φ and its derivatives vanish as $x\to\pm\infty$. Then, integrating Eq. (16) over the real line, and using Eqns. (17) and (18), we obtain

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\int_{-\infty}^{\infty}|\varphi(x,t)|^2\,dx &= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial x}\left(\varphi^*\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial\varphi^*}{\partial x}\right)\right]dx \\ &= -\frac{\hbar^2}{2m}\left(\varphi^*\frac{\partial\varphi}{\partial x}-\varphi\frac{\partial\varphi^*}{\partial x}\right)\Bigg|_{-\infty}^{\infty} \\ &= 0\,. \end{aligned} \tag{19}$$

Thus, the quantity $\int_{-\infty}^{\infty}|\varphi(x,t)|^2\,dx$ is independent of time for solutions of the free-particle Schrödinger equation which correspond to wave packets.

3. (a) The multiplication of two rational numbers, m/n and p/q, where m,n,p and q are integers, yields another rational number, mp/nq, so closure is obeyed. The unit is 1, multiplication is an associative operation, and the inverse of m/n is n/m, which is a rational number. The set excludes zero, so the problem of finding the inverse of zero does not arise. Hence, the rational numbers, excluding zero, under multiplication form a group.

   (b) The sum of two non-negative integers is a non-negative integer, thus ensuring closure, addition is associative, the unit is zero, but the inverse under addition of a negative integer n is $-n$, which is a *negative* integer and, therefore, excluded from the set. Hence, the non-negative integers under addition do not form a group.

(c) The sum of two even integers $2m$ and $2n$, where m and n are any two integers, is $2(m+n)$, which is an even integer, so closure is obeyed. Addition is associative, the unit is zero, which is an even integer, and the inverse of $2n$ is $-2n$, which is also an even integer. Hence, the even integers under addition form a group.

(d) The multiplication of two elements amounts to the addition of the integers $0, 1, \ldots, n-1$, modulo n, i.e., the addition of any two elements and, if the sum lies out side of this range, subtract n to bring it into the range. Thus, the multiplication of two nth roots of unity is again an nth root of unity, multiplication is associative, and the unit is 1. The inverse of $\mathrm{e}^{2\pi mi/n}$ is therefore

$$\mathrm{e}^{-2\pi mi/n} = \mathrm{e}^{2\pi i(n-m)/n} \tag{20}$$

Thus, the nth roots of unity form a group for any value of n.

(e) For the set of integers under ordinary subtraction, the difference between two integers n and m is another integer p, $n-m=p$, the identity is 0, since, $n-0=0$, and every integer is its own inverse, since $n-n=0$. However, subtraction is not associative because

$$(n-m)-p \neq n-(m-p) = n-m+p$$

Hence, the integers under ordinary subtraction do not form a group.

4. Let $u(x;\lambda)$ and $v(x;\lambda)$ be the fundamental solutions of the Liouville equation with the boundary conditions $y(a)=y(b)=0$. Then there are constants A and B which allow any solution y to be expressed as a linear combination of this fundamental set:

$$y(x;\lambda) = Au(x;\lambda) + Bv(x;\lambda) \tag{21}$$

These constants are determined by requiring $y(x;\lambda)$ to satisfy the boundary conditions given above. Applying these boundary conditions to the expression in (21) leads to the following equations:

$$\begin{aligned} y(a;\lambda) &= Au(a;\lambda) + Bv(a;\lambda) = 0 \\ y(b;\lambda) &= Au(b;\lambda) + Bv(b;\lambda) = 0 \end{aligned} \tag{22}$$

Equations (22) are two simultaneous equations for the unknown quantities A and B. To determine the conditions which guarantee that this system of equations has a nontrivial solution, we write these equations in matrix form:

$$\begin{pmatrix} u(a;\lambda) & v(a;\lambda) \\ u(b;\lambda) & v(b;\lambda) \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{23}$$

Thus, we see that Equations (22) can be solved for nonzero values of A and B only if the determinant of the matrix of coefficients in (23) vanishes. Otherwise, the only solution is $A = B = 0$, which yields the trivial solution $y = 0$. The condition for a nontrivial solution of (22) is, therefore, given by

$$\begin{vmatrix} u(a;\lambda) & v(a;\lambda) \\ u(b;\lambda) & v(b;\lambda) \end{vmatrix} = u(a;\lambda)v(b;\lambda) - u(b;\lambda)v(a;\lambda) = 0 \tag{24}$$

This guarantees that the solution for A and B is unique. Hence, the eigenvalues are non-degenerate.