Group Theory

Problem Set 2 October 16, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Show that, by requiring the existence of an identity in a group G, it is sufficient to require only a *left* identity, $ea = a$, or only a *right* identity $ae = a$, for every element a in G, since these two quantities *must* be equal.

2.* Similarly, show that it is sufficient to require only a *left* inverse, $a^{-1}a = e$, or only a *right* inverse $aa^{-1} = e$, for every element a in G, since these two quantities must also be equal.

3. Show that for any group G, $(ab)^{-1} = b^{-1}a^{-1}$.

4.* For the elements $g_1, g_2, \ldots, g_n$ of a group, determine the inverse of the n-fold product $g_1 g_2 \cdots g_n$.

5.* Show that a group is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$. You need to show that this condition is both necessary and sufficient for the group to be Abelian.

6. By explicit construction of multiplication tables, show that there are two distinct structures for groups of order 4. Are either of these groups Abelian?

7.* Consider the group of order 3 discussed in Section 2.4. Suppose we regard the rows of the multiplication table as individual permutations of the elements $\{e, a, b\}$ of this group. We label the permutations π_g by the group element corresponding to that row:

$$\pi_e = \begin{pmatrix} e & a & b \\ e & a & b \end{pmatrix}, \qquad \pi_a = \begin{pmatrix} e & a & b \\ a & b & e \end{pmatrix}, \qquad \pi_b = \begin{pmatrix} e & a & b \\ b & e & a \end{pmatrix}$$

(a) Show that, under the composition law for permutations discussed in Section 2.3, the multiplication table of the 3-element group is preserved by this association, e.g., $\pi_a \pi_b = \pi_e$.

(b) Show that for every element g in $\{e, a, b\}$,

$$\pi_g = \begin{pmatrix} e & a & b \\ g & ga & gb \end{pmatrix}$$

Hence, show that the π_g have the same multiplication table as the 3-element group.

(c) Determine the relationship between this group and S_3. This is an example of Cayley's theorem.

(d) To which of the operations on an equilateral triangle in Fig. 2.1 do these group elements correspond?