Group Theory

Problem Set 4 October 30, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Given a set of matrices $D(g)$ that form a representation a group G, show that the matrices which are obtainable by a similarity transformation $UD(g)U^{-1}$ are also a representation of G.

2.* Show that the trace of three matrices A, B, and C satisfies the following relation:

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$$

3. Generalize the result in Problem 4 to show that the trace of an n-fold product of matrices is invariant under cyclic permutations of the product.

4.* Show that the trace of an arbitrary matrix A is invariant under a similarity transformation UAU^{-1}.

5. Consider the following representation of S_3:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \qquad b = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

$$c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad d = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \qquad f = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$$

How can these matrices be permuted to provide an equally faithful representation of S_3? Relate your result to the class identified with each element.

6.* Consider the planar symmetry operations of an equilateral triangle. Using the matrices in Example 3.2 determined from transformations of the coordinates in Fig. 3.1, construct a *three*-dimensional representation of S_3 in the (x, y, z) coordinate system, where the z-axis emanates from the geometric center of the triangle. Is this representation reducible or irreducible? If it is reducible determine the irreducible representations which form the direct sum of this representation.

7. Show that two matrices are simultaneously diagonalizable if and only if they commute.

Hint: Two matrices A and B are simultaneously diagonalizable if the *same* similarity transformation brings both matrices into a form where they have only diagonal entries. Proving that simultaneous diagonalizability implies commutativity is straightforward. To prove the converse, suppose that there is a similarity transformation which brings *one* of the matrices into diagonal form. By writing out the matrix elements of the products and using the fact that A and B commute, show that the same similarity transformation *must* also diagonalize the other matrix.

8.* What does the result of Problem 7 imply about the dimensionalities of the irreducible representations of Abelian groups?

9.* Verify that the matrices

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$$

form a representation for the two-element group $\{e, a\}$. Is this representation reducible or irreducible? If it is reducible determine the one-dimensional representations which form the direct sum of this representation.

10.* Prove the relations in Eqns (3.9) and (3.11).