Group Theory

Solutions to Problem Set 4 November 9, 2001

1. Let $D'(g) = UD(g)U^{-1}$, where $D(g)$ is a representation of a group G with elements g. To show that $D'(g)$ is also a representation of G, it is sufficient to show that this representation preserves the multiplication table of G. Thus, let a and b be any two elements of G with matrix representations $D(a)$ and $D(b)$. The product ab is represented by

$$D(ab) = D(a)D(b)\,.$$

Therefore,

$$\begin{aligned} D'(ab) &= UD(ab)U^{-1} \\ &= UD(a)D(b)U^{-1} \\ &= UD(a)U^{-1}UD(b)U' \\ &= D'(a)D'(b)\,, \end{aligned}$$

so multiplication is preserved and $D'(g)$ is therefore also a representation of G.

2. The trace of a matrix is the sum of its diagonal elements. Therefore, the trace of the product of three matrices A, B, and C is given by

$$\mathrm{tr}(ABC) = \sum_{ijk} A_{ij}B_{jk}C_{ki}\,.$$

By using the fact that i, j, and k are dummy summation indices with the same range, this sum can be written in the equivalent forms

$$\sum_{ijk} A_{ij}B_{jk}C_{ki} = \sum_{ijk} C_{ki}A_{ij}B_{jk} = \sum_{ijk} B_{jk}C_{ki}A_{ij}\,.$$

But the second and third of these are

$$\sum_{ijk} C_{ki} A_{ij} B_{jk} = \mathrm{tr}(CAB)$$

and

$$\sum_{ijk} B_{jk} C_{ki} A_{ij} = \mathrm{tr}(BCA)\,,$$

respectively. Thus, we obtain the relation

$$\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)\,.$$

3. The trace of an n-fold product, $A_1 A_2 \cdots A_n$ is

$$\mathrm{tr}(A_1 A_2 \cdots A_n) = \sum_{i_1, i_2, \ldots i_n} (A_1)_{i_1 i_2} (A_2)_{i_2 i_3} \cdots (A_n)_{i_n i_1}\,.$$

Proceeding as in Problem 2, we observe that the i_k $(k = 1, \ldots, n)$ are dummy summation indices all of which have the same range. Thus, any cyclic permutation of the matrices in the product leaves the sum and, hence, the trace invariant.

4. From Problem 2, we have that

$$\mathrm{tr}(UAU^{-1}) = \mathrm{tr}(U^{-1}UA) = \mathrm{tr}(A)\,,$$

so a similarity transformation leaves the trace of a matrix invariant.

5. Given a faithful representation of a group, similarity transformations of the matrices provide equally faithful representations. Since we wish to obtain permutations of a particular matrix representation, we base our similarity transformations on the non-unit elements in the group. Thus, consider the following similarity transformations:

$$\begin{aligned} a\{e,a,b,c,d,f\}a^{-1} &= \{e,a,c,b,f,d\}\,, \\ b\{e,a,b,c,d,f\}b^{-1} &= \{e,c,b,a,f,d\}\,, \\ c\{e,a,b,c,d,f\}c^{-1} &= \{e,b,a,c,f,d\}\,, \\ d\{e,a,b,c,d,f\}d^{-1} &= \{e,b,c,a,d,f\}\,, \\ e\{e,a,b,c,d,f\}e^{-1} &= \{e,c,a,b,d,f\}\,. \end{aligned} \qquad (1)$$

Thus, the following permutations of the elements $\{e,a,b,c,d,f\}$ provide equally faithful representations:

$$\begin{aligned} &\{e,a,c,b,f,d\}\,, \quad \{e,c,b,a,f,d\}\,, \\ &\{e,b,a,c,f,d\}\,, \quad \{e,b,c,a,d,f\}\,, \\ &\{e,c,a,b,d,f\}\,. \end{aligned}$$

Notice that only elements within the same class can be permuted. For S_3, the classes are $\{e\}$, $\{a,b,c\}$, $\{d,f\}$.

6. In the basis (x,y,z) where x and y are given in Fig. 3.1 and the z-axis emanates from the origin of this coordinate system (the geometric center of the triangle), all of the symmetry operations of the equilateral triangle leave the z-axis invariant. This is because the z-axis is either an axis of rotation (for operations d and f) or lies within the reflection plane (for operations a, b, and c). Hence,

the matrices of these operations are given by

$$e=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\qquad a=\tfrac{1}{2}\begin{pmatrix}1&-\sqrt{3}&0\\-\sqrt{3}&-1&0\\0&0&2\end{pmatrix},$$

$$b=\tfrac{1}{2}\begin{pmatrix}1&\sqrt{3}&0\\\sqrt{3}&-1&0\\0&0&2\end{pmatrix},\qquad c=\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix},$$

$$d=\tfrac{1}{2}\begin{pmatrix}-1&-\sqrt{3}&0\\\sqrt{3}&-1&0\\0&0&2\end{pmatrix},\quad f=\tfrac{1}{2}\begin{pmatrix}-1&\sqrt{3}&0\\-\sqrt{3}&-1&0\\0&0&2\end{pmatrix}.$$

This representation is seen to be *reducible* and that it is the direct sum of the representation in Example 3.2 (which, as discussed in Example 3.4, is irreducible) and the identical representation.

7. We must show (i) that two matrices which are simultaneously diagonalizable commute and (ii) that two matrices which commute are simultaneously diagonalizable. Showing (i) is straightforward. For two $d\times d$ matrices A and B which are simultaneously diagonalizable, there is a matrix U such that

$$UAU^{-1}=D_A \qquad \text{and} \qquad UBU^{-1}=D_B\,,$$

where D_A and D_B are diagonal forms of these matrices. Clearly, therefore, we have that

$$D_AD_B=D_BD_A\,.$$

Hence, transforming back to the original basis,

$$\underbrace{(U^{-1}D_AU)}_{A}\underbrace{(U^{-1}D_BU)}_{B}=\underbrace{(U^{-1}D_BU)}_{B}\underbrace{(U^{-1}D_AU)}_{A}\,,$$

so A and B commute.

Now suppose that A and B commute and there is a transformation that brings *one* of these matrices, say A, into the diagonal form D_A:

$$UAU^{-1} = D_A \,.$$

Then, with

$$UBU^{-1} = B' \,,$$

the commutation relation $AB = BA$ transforms to

$$D_A B' = B' D_A \,.$$

The (i, j)th matrix element of these products is

$$\begin{aligned}(D_A B')_{ij} &= \sum_k (D_A)_{ik}(B')_{kj} = (D_A)_{ii}(B')_{ij} \\ = (B' D_A)_{ij} &= \sum_k (B')_{ik}(D_A)_{kj} = (B')_{ij}(D_A)_{jj} \,.\end{aligned}$$

After a simple rearrangement, we have

$$(B')_{ij}[(D_A)_{ii} - (D_A)_{jj}] = 0 \,.$$

There are three cases to consider:

Case I. All of the diagonal entries of D_A are distinct. Then,

$$(D_A)_{ii} - (D_A)_{jj} \neq 0 \qquad \text{if } i \neq j \,,$$

so *all* of the off-diagonal matrix elements of B' vanish, i.e., B' is a diagonal matrix. Thus, the same similarity transformation which diagonalizes A also diagonalizes B.

Case II. All of the diagonal entries of D_A are the same. In this case D_A is proportional to the unit matrix, $D_A = cI$, for some complex constant c. Hence, this matrix is *always* diagonal,

$$U(cI)U^{-1} = cI$$

and, in particular, it is diagonal when B is diagonal.

Case III. Some of the diagonal entries are the same and some are distinct. If we arrange the elements of D_A such that the first p elements are the same, $(D_A)_{11} = (D_A)_{22} = \cdots = (D_A)_{pp}$, then D_A has the general form

$$D_A = \begin{pmatrix} cI_p & 0 \\ 0 & D'_A \end{pmatrix},$$

where I_p is the $p \times p$ unit matrix and c is a complex constant. From Cases I and II, we deduce that B must be of the form

$$B = \begin{pmatrix} B_p & 0 \\ 0 & D'_B \end{pmatrix},$$

where B_p is some $p \times p$ matrix and D'_B is a diagonal matrix. Let V_p be the matrix which diagonalizes B:

$$V_p B_p V_p^{-1} = D''_B .$$

Then the matrix

$$V = \begin{pmatrix} V_p & 0 \\ 0 & I_{d-p} \end{pmatrix}$$

diagonalizes B while leaving D_A unchanged. Here, I_{d-p} is the $(d-p) \times (d-p)$ unit matrix.

Hence, in all three cases, we have shown that the same transformation which diagonalizes A also diagonalizes B.

8. The matrices of any representation $\{A_1, A_2, \ldots, A_n\}$ of an Abelian group G commute:

$$A_i A_j = A_j A_i$$

for all i and j. Hence, according to Problem 7, these matrices can all be simultaneously diagonalized. Since this is true of *all* representations of G, we conclude that all irreducible representations of Abelian groups are one-dimensional, i.e., they are numbers with ordinary multiplication as the composition law.

9. To verify that the matrices

$$e=\begin{pmatrix}1&0\\0&1\end{pmatrix},\qquad a=\tfrac{1}{2}\begin{pmatrix}-1&\sqrt{3}\\\sqrt{3}&1\end{pmatrix}\tag{2}$$

form a representation for the two-element group $\{e,a\}$, we need to demonstrate that the multiplication table for this group,

	e	a
e	e	a
a	a	e

is fulfilled by these matrices. The products $e^2=e$, $ea=a$, and $ae=a$ can be verified by inspection. The product a^2 is

$$a^2=\tfrac{1}{4}\begin{pmatrix}-1&\sqrt{3}\\\sqrt{3}&1\end{pmatrix}\begin{pmatrix}-1&\sqrt{3}\\\sqrt{3}&1\end{pmatrix}=\begin{pmatrix}1&0\\0&1\end{pmatrix}=e\,,$$

so the matrices in (2) form a representation of the two-element group.

Since these matrices commute, they can be diagonalized simultaneously (Problem 7). Since the matrix is the unit matrix, we can determine the diagonal form of a, simply by finding its eigenvalues. The characteristic equation of a is

$$\begin{aligned} det(a-\lambda I) &= \begin{vmatrix}-\frac{1}{2}-\lambda & \frac{1}{2}\sqrt{3}\\ \frac{1}{2}\sqrt{3} & \frac{1}{2}-\lambda\end{vmatrix}\\ &= -(\tfrac{1}{2}-\lambda)(\tfrac{1}{2}+\lambda)-\tfrac{3}{4}=\lambda^2-1\,. \end{aligned}$$

which yields $\lambda = \pm 1$. Therefore, diagonal form of a is

$$a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3}$$

so this representation is the direct sum of the identical representation $\{1, 1\}$, and the "parity" representation $\{1, -1\}$. Note that, according to Problem 8, *every* representation of the two-element group with dimensionality greater than two *must* be reducible.

10. The relations in (3.9) and (3.11) can be proven simultaneously, since they differ only by complex conjugation, which preserves the order of matrices. The (i, j)th matrix element of n-fold product of matrices $A_1, A_2, \ldots, A_n$ is

$$(A_1 A_2 \cdots A_n)_{ij} = \sum_{k_1, k_2, \ldots, k_{n-1}} (A_1)_{ik_1} (A_2)_{k_1 k_2} \cdots (A_n)_{k_{n-1} j} \,.$$

The corresponding matrix element of the transpose of this product is

$$[(A_1 A_2 \cdots A_n)^{\mathrm{t}}]_{ij} = (A_1 A_2 \cdots A_n)_{ji} \,.$$

Thus, since the k_i are dummy indices,

$$[(A_1 A_2 \cdots A_n)^{\mathrm{t}}]_{ij} = \sum_{k_1, k_2, \ldots, k_{n-1}} (A_1)_{jk_1} (A_2)_{k_1 k_2} \cdots (A_n)_{k_{n-1} i}$$

$$= \sum_{k_1, k_2, \ldots, k_{n-1}} (A_1^{\mathrm{t}})_{k_1 j} (A_2^{\mathrm{t}})_{k_2 k_1} \cdots (A_n^{\mathrm{t}})_{ik_{n-1}}$$

$$= \sum_{k_1, k_2, \ldots, k_{n-1}} (A_n^{\mathrm{t}})_{ik_{n-1}} (A_{n-1}^{\mathrm{t}})_{k_{n-1} k_{n-2}} \cdots (A_2^{\mathrm{t}})_{k_2 k_1} (A_1^{\mathrm{t}})_{k_1 j}$$

We conclude that

$$(A_1 A_2 \cdots A_n)^{\mathrm{t}} = A_n^{\mathrm{t}} A_{n-1}^{\mathrm{t}} \cdots A_1^{\mathrm{t}}$$

and, similarly, that

$$(A_1 A_2 \cdots A_n)^\dagger = A_n^\dagger A_{n-1}^\dagger \cdots A_1^\dagger$$