Group Theory

Solutions to Problem Set 5 November 16, 2001

1. In proving Theorem 3.2, we established that $B_iB_i^\dagger = I$, where

$$B_i = D^{-1/2}\tilde{A}_iD^{1/2}$$

To show that this result implies that $B_i^\dagger B_i = I$, we first use the definitions of B_i and $B_i^\dagger$ to write

$$B_i^\dagger B_i = D^{1/2}\tilde{A}_i^\dagger D^{-1}\tilde{A}_iD^{1/2}$$

We can find an expression for D^{-1} by first rearranging

$$B_iB_i^\dagger = D^{-1/2}\tilde{A}_iD\tilde{A}_i^\dagger D^{-1/2} = I$$

as

$$\tilde{A}_iD\tilde{A}_i^\dagger = D$$

Then, taking the inverse of both sides of this equation yields

$$D^{-1} = \tilde{A}_i^{\dagger-1}D^{-1}\tilde{A}_i^{-1}$$

Therefore,

$$\begin{aligned} B_i^\dagger B_i &= D^{1/2}\tilde{A}_i^\dagger D^{-1}\tilde{A}_iD^{1/2} \\ &= D^{1/2}\tilde{A}_i^\dagger(\tilde{A}_i^{\dagger-1}D^{-1}\tilde{A}_i^{-1})\tilde{A}_iD^{1/2} \\ &= I \end{aligned}$$

2. (a) The multiplication table for the three-element group is shown below (Sec. 2.4):

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

We can see immediately that $a^2 = b$ and that $ab = ba = a^3 = e$, Thus, the three-element group can be written as $\{a, a^2, a^3 = e\}$, i.e., it is a cyclic group (and, therefore, Abelian).

(b) By choosing a one-dimensional representation $a = z$, for some complex number z, the multiplication table requires that $a^3 = e$, which means that $z^3 = 1$.

(c) There are three solutions to $z^3 = 1$: $z = 1, \mathrm{e}^{2\pi i/3}, and \mathrm{e}^{4\pi i/3}$. The three irreducible representations are obtained by choosing $a = 1$, $a = \mathrm{e}^{2\pi i/3}$, and $a = \mathrm{e}^{4\pi i/3}$. Denoting these representations by Γ_1, Γ_2, and Γ_3, we obtain

	e	a	b
Γ_1	1	1	1
Γ_2	1	$\mathrm{e}^{2\pi i/3}$	$\mathrm{e}^{4\pi i/3}$
Γ_2	1	$\mathrm{e}^{4\pi i/3}$	$\mathrm{e}^{2\pi i/3}$

3. The preceding Problem can be generalized to any cyclic group of order n. The elements of this group are $\{g, g^2, \ldots, g^n = e\}$. By writing $g = z$, we require that $z^n = 1$. The solutions to this equation are the nth roots of unity:

$$z = \mathrm{e}^{2m\pi i/n}, \quad m = 0, 1, 2, \ldots, n-1$$

Accordingly, there are n irreducible representations based on the n choices $z = \mathrm{e}^{2m\pi i/n}$ together with the requirements of the group multiplication table.

4. Suppose that we have a representation of an Abelian group of dimensionality d is greater than one. Suppose furthermore that these matrices are not all unit matrices (for, if they were, the representation would already be reducible to the d-fold direct sum of the identical representation.) Then, since the group is Abelian, and the representation must reflect this fact, *any* non-unit matrix in the representation commutes with all the other matrices in the representation. According to Schur's First Lemma, this renders the representation reducible. Hence, all the irreducible representations of an Abelian group are one-dimensional.

5. For the following matrices,

$$e = d = f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = b = c = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

to be a representation of S_3, their products must preserve the multiplication table of this group, which was discussed in Sec. 2.4 and is displayed below:

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	f	e	d	c	a
c	c	d	f	e	a	b
d	d	c	a	b	f	e
f	f	b	c	a	e	d

To determine the multiplication table of this representation, we use the notation E for the unit matrix, corresponding to the elements e, d, and f, and A for the matrix corresponding to the elements a, b, and c. Then, by observing that $A^2 = E$ (as is required by the group multiplication table) the multiplication table of this representation is straightforward to calculate, and is shown below:

	e	a	b	c	d	f
e	E	A	A	A	E	E
a	A	E	E	E	A	A
b	A	E	E	E	A	A
c	A	E	E	E	A	A
d	E	A	A	A	E	E
f	E	A	A	A	E	E

If we now take the multiplication table of S_3 and perform the mapping $\{e, d, f\} \mapsto E$ and $\{a, b, c\} \mapsto A$, we get the same table as that just obtained by calculating the matrix products directly. Hence, these matrices form a representation of S_3.

From Schur's First Lemma, we see that the matrix

$$\tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

commutes with all the matrices of the representation. Since this is not a unit matrix, the representation must be reducible.

The diagonal form of these matrices must have entries of one-dimensional irreducible representations. Two one-dimensional irreducible representations of S_3 (we will see later that these are the *only* irreducible representations of S_3) are (Example 3.2) the identical representation,

$$A_e = 1, \quad A_a = 1, \quad A_b = 1,$$

$$A_c = 1, \quad A_d = 1, \quad A_f = 1$$

and the 'parity' representation,

$$A_e = 1, \quad A_a = -1, \quad A_b = -1,$$

$$A_c = -1, \quad A_d = 1, \quad A_f = 1$$

Since the diagonal forms of the matrices are obtained by performing a similarity transformation on the original matrices, which preserves the trace, they must take the form

$$e = d = f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad a = b = c = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

i.e., this reducible representation contains both the identical and parity representations.