Group Theory

Problem Set 6 November 13, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1.* Verify the Great Orthogonality Theorem for the following irreducible representation of S_3:

$$e=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad a=\tfrac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}, \qquad b=\tfrac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$$

$$c=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \qquad d=\tfrac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}, \qquad f=\tfrac{1}{2}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$$

2.* Does the following representation of the three-element group $\{e,a,b\}$:

$$e=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad a=\tfrac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix}, \qquad b=\tfrac{1}{2}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix}$$

satisfy the Great Orthogonality Theorem? Explain your answer.

3.* Specialize the Great Orthogonality Theorem to Abelian groups. When viewed as the components of a vector in a $|G|$-dimensional space, what does the Great Orthogonality Theorem state about the relationship between different irreducible representations? What bound does this place on the number of irreducible representations of an Abelian group?

4.* Consider the irreducible representations of the three-element calculated in Problem 2 of Problem Set 5.

(a) Verify that the Great Orthogonality Theorem, in the reduced form obtained in Problem 3, is satisfied for these representations.

(b) In view of the discussion in Sec. 4.4, would you expect to find any other irreducible representations of this group?

(c) Would you expect your answer in (b) to apply to cyclic groups of any order?

5.* Consider any Abelian group. By using the notion of the order of an element (Sec. 2.4), determine the *magnitude* of every element in a representation. Is this consistent with the Great Orthogonality Theorem?