Group Theory

Solutions to Problem Set 6 November 23, 2001

1. The Great Orthogonality Theorem states that, for the matrix elements of the same irreducible representation $\{A_1, A_2, \ldots, A_{|G|}\}$ of a group G,

$$\sum_\alpha (A_\alpha)_{ij}(A_\alpha)^*_{i'j'} = \frac{|G|}{d}\delta_{i,i'}\delta_{j,j'}\,.$$

Thus, we first form the vectors $\boldsymbol{V}_{ij}$ whose components are the (i,j)th elements taken from each matrix in the representation in some fixed order. The Great Orthogonality Theorem can then be expressed more concisely as

$$\boldsymbol{V}_{ij}\cdot\boldsymbol{V}^*_{i'j'} = \frac{|G|}{d}\delta_{i,i'}\delta_{j,j'}\,.$$

For the given representation of S_3,

$$e=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\qquad a=\tfrac{1}{2}\begin{pmatrix}1 & -\sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix},\qquad b=\tfrac{1}{2}\begin{pmatrix}1 & \sqrt{3}\\ \sqrt{3} & -1\end{pmatrix},$$

$$c=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\quad d=\tfrac{1}{2}\begin{pmatrix}-1 & \sqrt{3}\\ -\sqrt{3} & -1\end{pmatrix},\qquad f=\tfrac{1}{2}\begin{pmatrix}-1 & -\sqrt{3}\\ \sqrt{3} & -1\end{pmatrix},$$

these vectors are:

$$\boldsymbol{V}_{11} = \left(1, \tfrac{1}{2}, \tfrac{1}{2}, -1, -\tfrac{1}{2}, -\tfrac{1}{2}\right),$$

$$\boldsymbol{V}_{12} = \left(0, -\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}, 0, \tfrac{1}{2}\sqrt{3}, -\tfrac{1}{2}\sqrt{3}\right),$$

$$\boldsymbol{V}_{21} = \left(0, -\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}, 0, -\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}\right),$$

$$\boldsymbol{V}_{22} = \left(1, -\tfrac{1}{2}, -\tfrac{1}{2}, 1, -\tfrac{1}{2}, -\tfrac{1}{2}\right).$$

Note that, since all of the entries are real, complex conjugation is not required for substitution into the Great Orthogonality Theorem. For $i = j$ and $i' = j'$, with $|G| = 6$ and $d = 2$, we have

$$\begin{aligned}
\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{11} &= 1 + \tfrac{1}{4} + \tfrac{1}{4} + 1 + \tfrac{1}{4} + \tfrac{1}{4} = 3\,, \\
\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{12} &= 0 + \tfrac{3}{4} + \tfrac{3}{4} + 0 + \tfrac{3}{4} + \tfrac{3}{4} = 3\,, \\
\boldsymbol{V}_{21} \cdot \boldsymbol{V}_{21} &= 0 + \tfrac{3}{4} + \tfrac{3}{4} + 0 + \tfrac{3}{4} + \tfrac{3}{4} = 3\,. \\
\boldsymbol{V}_{22} \cdot \boldsymbol{V}_{22} &= 1 + \tfrac{1}{4} + \tfrac{1}{4} + 1 + \tfrac{1}{4} + \tfrac{1}{4} = 3\,,
\end{aligned}$$

all of which are in accord with the Great Orthogonality Theorem. For $i \neq j$ and/or $i' \neq j'$, we have

$$\begin{aligned}
\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{12} &= 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} + 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} = 0\,, \\
\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{21} &= 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} + 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} = 0\,, \\
\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{22} &= 1 - \tfrac{1}{4} - \tfrac{1}{4} - 1 + \tfrac{1}{4} + \tfrac{1}{4} = 0\,, \\
\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{21} &= 0 + \tfrac{3}{4} + \tfrac{3}{4} + 0 - \tfrac{3}{4} - \tfrac{3}{4} = 0\,, \\
\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{22} &= 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} + 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} = 0\,, \\
\boldsymbol{V}_{21} \cdot \boldsymbol{V}_{22} &= 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} + 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} = 0\,,
\end{aligned}$$

which is also in accord with the Great Orthogonality Theorem.

2. For the following two-dimensional representation of the three-element group,

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad a = \tfrac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}, \quad b = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix},$$

we again form the vectors $\boldsymbol{V}_{ij}$ whose components are the (i, j)th elements of each matrix in the representation:

$$\boldsymbol{V}_{11} = \left(1, -\tfrac{1}{2}, -\tfrac{1}{2}\right),$$

$$\boldsymbol{V}_{12} = \left(0, \tfrac{1}{2}\sqrt{3}, -\tfrac{1}{2}\sqrt{3}\right),$$

$$\boldsymbol{V}_{21} = \left(0, -\tfrac{1}{2}\sqrt{3}, \tfrac{1}{2}\sqrt{3}\right),$$

$$\boldsymbol{V}_{22} = \left(1, -\tfrac{1}{2}, -\tfrac{1}{2}\right).$$

Calculating the summation in the Great Orthogonality Theorem, first with $i = j$ and $i' = j'$, we have

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{11} = 1 + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{2},$$

$$\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{12} = 0 + \tfrac{3}{4} + \tfrac{3}{4} = \tfrac{3}{2},$$

$$\boldsymbol{V}_{21} \cdot \boldsymbol{V}_{12} = 0 + \tfrac{3}{4} + \tfrac{3}{4} = \tfrac{3}{2},$$

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{11} = 1 + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{2},$$

all of which are in accord with the Great Orthogonality Theorem with $|G| = 3$ and $d = 2$. Performing the analogous summations with $i \neq j$ and/or $i' \neq j'$, yields

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{12} = 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} = 0,$$

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{21} = 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} = 0,$$

$$\boldsymbol{V}_{11} \cdot \boldsymbol{V}_{22} = 1 + \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{3}{2},$$

$$\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{21} = 0 - \tfrac{3}{4} - \tfrac{3}{4} = -\tfrac{3}{2},$$

$$\boldsymbol{V}_{12} \cdot \boldsymbol{V}_{22} = 0 - \tfrac{1}{4}\sqrt{3} + \tfrac{1}{4}\sqrt{3} = 0,$$

$$\boldsymbol{V}_{21} \cdot \boldsymbol{V}_{22} = 0 + \tfrac{1}{4}\sqrt{3} - \tfrac{1}{4}\sqrt{3} = 0,$$

which is *not* consistent with the Great Orthogonality Theorem, since *all* of these quantities must vanish. If there is even a single violation of the Great Orthogonality Theorem, as is the case here, the representation is necessarily reducible.

3. All of the irreducible representations of an Abelian group are one-dimensional (e.g., Problem 4, Problem Set 5). Hence, for Abelian groups, the Great Orthogonality Theorem reduces to

$$\sum_\alpha A_\alpha^k A_\alpha^{k'*} = |G|\delta_{k,k'}\,.$$

If we view the irreducible representations as $|G|$-dimensional vectors $\boldsymbol{A}^k$ with entries A_α^k,

$$\boldsymbol{A}^k = (A_1^k, A_2^k, \ldots A_{|G|}^k)\,,$$

then the Great Orthogonality Theorem can be written as a "dot" product:

$$\boldsymbol{A}^k\cdot\boldsymbol{A}^{k'*} = |G|\delta_{k,k'}\,.$$

This states that the irreducible representations of an Abelian group are *orthogonal* vectors in this $|G|$-dimensional space. Since there can be at most $|G|$ such vectors, the number of irreducible representations of an Abelian group is less than or equal to the order of the group.

4. (a) From Problem 2 of Problem Set 5, the irreducible representations of the three element group are:

	e	a	b
Γ_1	1	1	1
Γ_2	1	$\mathrm{e}^{2\pi i/3}$	$\mathrm{e}^{4\pi i/3}$
Γ_2	1	$\mathrm{e}^{4\pi i/3}$	$\mathrm{e}^{2\pi i/3}$

In the notation of Problem 3, we have

$$\boldsymbol{A}^1 = (1,1,1), \quad \boldsymbol{A}^2 = (1, \mathrm{e}^{2\pi i/3}, \mathrm{e}^{4\pi i/3}), \quad \boldsymbol{A}^3 = (1, \mathrm{e}^{4\pi i/3}, \mathrm{e}^{2\pi i/3})\,.$$

Note that some of these entries are complex. Thus, the distinct inner products between these vectors are

$$\boldsymbol{A}^1\cdot\boldsymbol{A}^{1*} = 1+1+1 = 3\,,$$

$$\begin{aligned}
\boldsymbol{A}^2\cdot\boldsymbol{A}^{2*} &= 1+1+1=3\,,\\
\boldsymbol{A}^3\cdot\boldsymbol{A}^{3*} &= 1+1+1=3\,,\\
\boldsymbol{A}^1\cdot\boldsymbol{A}^{2*} &= 1+\mathrm{e}^{-2\pi i/3}+\mathrm{e}^{-4\pi i/3}=0\,,\\
\boldsymbol{A}^1\cdot\boldsymbol{A}^{3*} &= 1+\mathrm{e}^{-4\pi i/3}+\mathrm{e}^{-2\pi i/3}=0\,,\\
\boldsymbol{A}^2\cdot\boldsymbol{A}^{3*} &= 1+\mathrm{e}^{-2\pi i/3}+\mathrm{e}^{2\pi i/3}=0\,,
\end{aligned}$$

all of which are consistent with the Great Orthogonality Theorem.

(b) In view of the fact that there are 3 mutually orthogonal vectors, there can be no additional irreducible representations of this group.

(c) For cyclic groups of order $|G|$, we determined that the irreducible representations were based on the $|G|$th roots of unity (Problem 3, Problem Set 5). Since this produces $|G|$ distinct irreducible representations, our procedure yields *all* of the irreducible representations of any cyclic group.

5. Every irreducible representation of an Abelian group is one-dimensional. Moreover, since every one of these representations is either a homomorphism or isomorphism of the group, with the operation in the representation being ordinary multiplication, the identity always corresponds to unity (Problem 9, Problem Set 3). Now, the order n of a group element g is the smallest integer for which

$$g^n = e\,.$$

For every element in any group $1 \le n \le |G|$. This relationship *must* be preserved by the irreducible representation. Thus, if A_g^k is the entry corresponding to the element g in the kth irreducible representation, then

$$(A_g^k)^n = 1\,,$$

i.e., A_g^k is the nth root of unity:

$$A_g^k = \mathrm{e}^{2m\pi i/n}, \quad m = 0, 1, \ldots, n-1\,.$$

The modulus of each of these quantities is clearly unity, so the modulus of *every* entry in the irreducible representations of an Abelian group is unity.

This is consistent with the Great Orthogonality Theorem when applied to a given representation (cf. Problem 3):

$$\sum_\alpha A_\alpha^k A_\alpha^{k^*} = \sum_\alpha \left| A_\alpha^k \right|^2 = |G|\,. \tag{1}$$