Group Theory

Problem Set 8 November 27, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

1. Show that if two matrices A and B are orthogonal, then their direct product $A \otimes B$ is also an orthogonal matrix.

2. Show that the trace of the direct product of two matrices A and B is the product of the traces of A and B:

$$\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$$

3.* Show that the direct product of groups G_a and G_b with elements $G_a = \{e, a_2, \ldots, a_{|G_a|}\}$ and $G_b = \{e, b_2, \ldots, b_{|G_b|}\}$, such that $a_i b_j = b_j a_i$ for all i and j, is a group. What is the order of this group?

4.* Use the Great Orthogonality Theorem to show that the direct product of irreducible representations of two groups is an irreducible representation of the direct product of those groups.

5.* For an n-fold degenerate set of eigenfunctions φ_i, $i, 1, 2, \ldots, n$, we showed show that the matrices $\Gamma(R_\alpha)$ generated by the group of the Hamiltonian,

$$R_\alpha \varphi_i = \sum_{j=1}^{n} \varphi_j \Gamma_{ji}(R_\alpha)$$

form a representation of that group. Show that if the φ_j are chosen to be an orthonormal set of functions, then this representation is *unitary*.

6.* The set of distinct functions obtained from a given function φ_i by operations in the group of the Hamiltonian, $\varphi_j = R_\alpha \varphi_i$, are called **partners**. Use the Great Orthogonality Theorem to show that two functions which belong to different irreducible representations or are different partners in the same unitary representation are orthogonal.

7. Consider a particle of mass m confined to a square in two dimensions whose vertices are located at $(1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$. The potential is taken to be zero within the square and infinite at the edges of the square. The eigenfunctions φ are of the form

$$\varphi_{p,q}(x, y) \propto \left\{ \begin{array}{c} \cos(k_p x) \\ \sin(k_p x) \end{array} \right\} \left\{ \begin{array}{c} \cos(k_q y) \\ \sin(k_q y) \end{array} \right\}$$

where $k_p = \frac{1}{2}p\pi$, $k_q = \frac{1}{2}q\pi$, and p and q are positive integers. The notation above means that $\cos(k_p x)$ is taken if p is odd, $\sin(k_p x)$ is taken if p is even, and similarly for the other factor. The corresponding eigenvalues are

$$E_{p,q} = \frac{\hbar^2\pi^2}{8m}(p^2 + q^2)$$

(a) Determine the eight planar symmetry operations of a square. These operations form the group of the Hamiltonian for this problem. Assemble the symmetry operations into equivalence classes.

(b) Determine the number of irreducible representations and their dimensions for this group. Do these dimensions appear to be broadly consistent with the degeneracies of the energy eigenvalues?

(c) Determine the action of each group operation on (x, y).

Hint: This can be done by inspection.

(d) Determine the characters corresponding to the identical, parity, and coordinate representations. Using appropriate orthogonality relations, complete the character table for this group.

(e) For which irreducible representations do the eigenfunctions $\varphi_{1,1}(x, y)$ and $\varphi_{2,2}(x, y)$ form bases?

(f) For which irreducible transformation do the eigenfunctions $\varphi_{1,2}(x, y)$ and $\varphi_{2,1}(x, y)$ form a basis?

(g) What is the degeneracy corresponding to $(p = 6, q = 7)$ and $(p = 2, q = 9)$? Is this a normal or accidental degeneracy?

(h) Are there eigenfunctions which form a basis for each of the irreducible representations of this group?

8.* Consider the regular hexagon in Problem Set 7. Suppose there is a vector perturbation, i.e., a perturbation that transforms as (x, y, z). Determine the selection rule associated with an initial state that transforms according to the "parity" representation.

Hint: The reasoning for determining the irreducible representations associated with (x, y, z) is analogous to that used in Section 6.6.2 for the equilateral triangle.