# Group Theory

Problem Set 9 December 4, 2001

Note: Problems marked with an asterisk are for Rapid Feedback.

**1.*** Consider the group $\mathrm{O}(n)$, the elements of which preserve the Euclidean length in $n$ dimensions:

$$x_1'^2 + x_2'^2 + \cdots + x_n'^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \,.$$

Show that these transformations have $\frac{1}{2}n(n-1)$ free parameters.

**2.** The condition that the Euclidean length is preserved in two dimensions, $x'^2 + y'^2 = x^2 + y^2$, was shown in lectures to require that

$$a_{11}^2 + a_{21}^2 = 1, \qquad a_{11}a_{12} + a_{21}a_{22} = 0, \qquad a_{12}^2 + a_{22}^2 = 1 \,.$$

Show that these requirements imply that

$$(a_{11}a_{22} - a_{12}a_{21})^2 = 1 \,.$$

**3.** Rotations in two dimensions can be parametrized by

$$R(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} .$$

Show that

$$R(\varphi_1 + \varphi_2) = R(\varphi_1)R(\varphi_2)$$

and, hence, deduce that this group is Abelian.

**4.*** We showed in lectures that a rotation $R(\varphi)$ by an angle $\varphi$ in two dimensions can be written as

$$R(\varphi) = \mathrm{e}^{\varphi X} \,,$$

where

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Verify that

$$\mathrm{e}^{\varphi X} = I\cos\varphi + X\sin\varphi \,,$$

where $I$ is the two-dimensional unit matrix. This shows that $\mathrm{e}^{\varphi X}$ is the rotation matrix in two dimensions.

**5.*** Consider the two-parameter group

$$x' = ax + b$$

Determine the infinitesimal operators of this group.

**6.*** Consider the group $C_{\infty v}$ which contains, in addition to all two-dimensional rotations, a reflection plane, denoted by $\sigma_v$ in, say, the $x$-$z$ plane. Is this group Abelian? What are the equivalence classes of this group?

Hint: Denoting reflection in the $x$–$z$ plane by $S$, show that $SR(\varphi)S^{-1} = R(-\varphi)$.

**7.** By extending the procedure used in lectures for SO(3), show that the infinitesimal generators of SO(4), the group of proper rotations in four dimensions which leave the quantity $x^2 + y^2 + z^2 + w^2$ invariant, are

$$A_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \qquad A_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \qquad A_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

$$B_1 = x\frac{\partial}{\partial w} - w\frac{\partial}{\partial x}, \qquad B_2 = y\frac{\partial}{\partial w} - w\frac{\partial}{\partial y}, \qquad B_3 = z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}$$

**8.** Show that the commutators of the generators obtained in Problem 7 are

$$[A_i, A_j] = \varepsilon_{ijk}A_k, \qquad [B_i, B_j] = \varepsilon_{ijk}A_k, \qquad [A_i, B_j] = \varepsilon_{ijk}B_k$$

**9.** Show that by making the linear transformation of the generators in Problem 7 to

$$J_i = \tfrac{1}{2}(A_i + B_i), \qquad K_i = \tfrac{1}{2}(A_i - B_i)$$

the commutators become

$$[J_i, J_j] = \varepsilon_{ijk}J_k, \qquad [K_i, K_j] = \varepsilon_{ijk}K_k, \qquad [J_i, K_j] = 0$$

This shows that locally SO(4)=SO(3)$\otimes$SO(3).