Group Theory

Solutions to Problem Set 9 December 14, 2001

1. The Lie group GL(n,R) has n^2 parameters, because the transformations can be represented as $n \times n$ matrices (with real entries). The requirement that the Euclidean length dimensions be preserved by such a transformation leads to the requirement that,

$$x_1'^2 + x_2'^2 + \cdots + x_n'^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \,.$$

Proceeding as in Sec. 7.2, we note that there are n conditions from the requirement that the coefficients of x_i, $i = 1, 2, \ldots, n$ be equal to unity. Then, there are $\frac{1}{2}n(n-1)$ conditions from the requirement that the coefficients of the *unique* products $x_i x_j$, $i \neq j$ vanish. Thus, beginning with n free parameters for GL(n,R), there are

$$n^2 - n - \tfrac{1}{2}n(n-1) = n^2 - n - \tfrac{1}{2}n^2 + \tfrac{1}{2}n = \tfrac{1}{2}n(n-1)$$

free parameters for O(n).

2. Beginning with the three conditions

$$a_{11}^2 + a_{21}^2 = 1, \qquad a_{11}a_{12} + a_{21}a_{22} = 0, \qquad a_{12}^2 + a_{22}^2 = 1\,,$$

we take the product of the first by the third equations and subtract the square of the second equation to obtain

$$\begin{aligned}
&(a_{11}^2 + a_{21}^2)(a_{12}^2 + a_{22}^2) - (a_{11}a_{12} + a_{21}a_{22})^2 \\
&= a_{11}^2 a_{12}^2 + a_{11}^2 a_{22}^2 + a_{21}^2 a_{12}^2 + a_{21}^2 a_{22}^2 - a_{11}^2 a_{12}^2 \\
&\qquad -2a_{11}a_{12}a_{21}a_{22} - a_{21}^2 a_{22}^2 \\
&= a_{11}^2 a_{22}^2 + a_{21}^2 a_{12}^2 - 2a_{11}a_{12}a_{21}a_{22} \\
&= (a_{11}a_{22} - a_{12}a_{21})^2 \\
&= 1\,.
\end{aligned}$$

Thus, the three constraints for orthogonal groups in two dimensions imply that the square of the determinant of such transformation must be equal to unity.

3. Forming the product of the the matrices corresponding to $R(\varphi_1)$ and $R(\varphi_2)$ yields

$$\begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \begin{pmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 & -\cos\varphi_1\sin\varphi_2 - \sin\varphi_1\cos\varphi_2 \\ \sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2 & -\sin\varphi_1\sin\varphi_2 + \cos\varphi_1\cos\varphi_2 \end{pmatrix}.$$

By invoking the standard trigonometric identities for the sines and cosines of the sum and difference of two angles,

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y\,,$$
$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y\,,$$

we can write

$$\begin{pmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{pmatrix} \begin{pmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{pmatrix}$$

$$= \begin{bmatrix} \cos(\varphi_1+\varphi_2) & -\sin(\varphi_1+\varphi_2) \\ \sin(\varphi_1+\varphi_2) & \cos(\varphi_1+\varphi_2) \end{bmatrix}.$$

Thus,

$$R(\varphi_1+\varphi_2) = R(\varphi_1)R(\varphi_2)\,.$$

4. The expression

$$R(\varphi) = \mathrm{e}^{\varphi X}\,,$$

where

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\,,$$

is defined by its Taylor series:

$$\mathrm{e}^{\varphi X} = \sum_{n=0}^{\infty} \frac{1}{n!}(\varphi X)^n\,. \tag{1}$$

Successive powers of X yield

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad X^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\,,$$

$$X^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\,,$$

whereupon this sequence is repeated. We can write this sequence in matrix form as $X^2 = -I, X^3 = -X, X^4 = I, \ldots$, where I is the 2×2 unit matrix. The powers of X are therefore given by

$$X^{2n} = \begin{cases} I, & n \text{ even} \\ -I, & n \text{ odd} \end{cases}$$

for even powers and

$$X^{2n+1} = \begin{cases} X, & n \text{ even} \\ -X, & n \text{ odd} \end{cases}$$

for odd powers Thus, the Taylor series in (1) may be written as

$$\mathrm{e}^{\varphi X} = \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\varphi^{2n}}_{\cos\varphi} I + \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\varphi^{2n+1}}_{\sin\varphi} X$$

$$= I\cos\varphi + X\sin\varphi$$

$$= \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix},$$

which is the rotation matrix in two dimensions.

5. The two parameter group

$$x' = ax + b$$

was discussed in Example 7.1. The identity was found to correspond to the parameters $a = 1$ and $b = 0$. The infinitesimal transformations are therefore given by

$$x' = (1 + \mathrm{d}a)x + \mathrm{d}b = x + x\,\mathrm{d}a + \mathrm{d}b\,.$$

If we substitute this into some function $f(x)$ and expand to first order in the parameters a and b, we obtain

$$f(x') = f(x + x\,\mathrm{d}a + \mathrm{d}b) = f(x) + x\frac{\partial f}{\partial x}\,\mathrm{d}a + \frac{\partial f}{\partial x}\,\mathrm{d}b\,,$$

from which we identify the infinitesimal operators

$$X_1 = x\frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial x}\,.$$

6. The group $C_{\infty v}$ contains all two-dimensional rotations and a vertical reflection plane, denoted by σ_v, in the x-z plane. Since this reflection changes the parity of the coordinate system, it changes the sense of the rotation angle φ. Thus, a rotation by φ in the original coordinate system corresponds to a rotation by $-\varphi$ in the

transformed coordinate system. Denoting the reflection operator by S, we then must have that

$$SR(\varphi)S^{-1} = R(-\varphi)\,. \tag{2}$$

Since $S = S^{-1}$, we can see this explicitly for the two-dimensional rotation matrix $R(\varphi)$:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix},$$

where the matrix on the right-hand side of this equation is $R(-\varphi)$. Equation (2) shows that (i) the group is no longer Abelian, and (ii) the equivalence classes correspond to rotations by φ and $-\varphi$.

7. Proceeding as in Section 7.4, the infinitesimal rotations in four dimensions which leave the quantity $x^2 + y^2 + z^2 + w^2$ invariant are

$$\begin{pmatrix} x' \\ y' \\ z' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & \varphi_1 & \varphi_2 & \varphi_3 \\ -\varphi_1 & 1 & \varphi_4 & \varphi_5 \\ -\varphi_2 & -\varphi_4 & 1 & \varphi_6 \\ -\varphi_3 & -\varphi_5 & -\varphi_6 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}.$$

Substituting this coordinate transformation into a differentiable function $F(x, y, z, w)$,

$$\begin{aligned} &F(x', y', z', w') \\ &= F(x + \varphi_1 y + \varphi_2 z + \varphi_3 w, y - \varphi_1 x - \varphi_4 z + \varphi_5 w, \\ &\qquad z - \varphi_2 x + \varphi_4 y + \varphi_6 w, w - \varphi_3 x - \varphi_5 y - \varphi_6 z)\,. \end{aligned}$$

and expanding to first order in the φ_i, yields

$$F(x', y', z', w') = F(x, y, z, w) + \varphi_1\Big(y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y}\Big)$$

$$= \varphi_2\Big(z\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial z}\Big) + \varphi_3\Big(w\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial w}\Big)$$
$$= \varphi_4\Big(y\frac{\partial F}{\partial z} - z\frac{\partial F}{\partial y}\Big) + \varphi_5\Big(w\frac{\partial F}{\partial y} - y\frac{\partial F}{\partial w}\Big)$$
$$= \varphi_6\Big(w\frac{\partial F}{\partial z} - z\frac{\partial F}{\partial w}\Big) .$$

From these equations and, if necessary, a change in sign of the corresponding φ_i, we can identify the following differential operators

$$A_1 = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} , \qquad A_2 = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x} , \qquad A_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} ,$$
$$B_1 = x\frac{\partial}{\partial t} - t\frac{\partial}{\partial x} , \qquad B_2 = y\frac{\partial}{\partial t} - t\frac{\partial}{\partial y} , \qquad B_3 = z\frac{\partial}{\partial t} - t\frac{\partial}{\partial z} .$$

8. With the infinitesimal generators calculated in Problem 7, we determine the commutators in the standard fashion. For the commutators between the A_i, we have

$$[A_1, A_2]f = A_1(A_2 f) - A_2(A_1 f)$$
$$= \Big(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\Big)\Big(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\Big) - \Big(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\Big)\Big(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\Big)$$
$$= xz\frac{\partial^2 f}{\partial y \partial z} - z^2\frac{\partial^2 f}{\partial y \partial x} - xy\frac{\partial^2 f}{\partial z^2} + y\frac{\partial f}{\partial x} + yz\frac{\partial^2 f}{\partial z \partial x}$$
$$-x\frac{\partial f}{\partial y} - xz\frac{\partial^2 f}{\partial z \partial y} + xy\frac{\partial^2 f}{\partial z^2} + z^2\frac{\partial^2 f}{\partial x \partial y} + yz\frac{\partial^2 f}{\partial x \partial z}$$
$$= y\frac{\partial f}{\partial x} - -x\frac{\partial f}{\partial y}$$
$$= A_3 f ,$$

$$[A_1, A_3]f = A_1(A_3 f) - A_3(A_1 f)$$

$$= \Big(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\Big)\Big(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\Big) - \Big(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\Big)\Big(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\Big)$$

$$= z\frac{\partial f}{\partial x} + yz\frac{\partial^2 f}{\partial y\partial z} - xz\frac{\partial^2 f}{\partial y^2} - y^2\frac{\partial^2 f}{\partial z\partial x} + xy\frac{\partial^2 f}{\partial z\partial y}$$

$$-yz\frac{\partial^2 f}{\partial x\partial y} + y^2\frac{\partial^2 f}{\partial x\partial z} + xz\frac{\partial^2 f}{\partial y^2} - x\frac{\partial f}{\partial z} - xy\frac{\partial^2 f}{\partial y\partial z}$$

$$= z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}$$

$$= -A_2 f\,,$$

$$[A_2, A_3]f = A_2(A_3 f) - A_3(A_2 f)$$

$$= \Big(x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}\Big)\Big(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\Big) - \Big(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\Big)\Big(x\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial x}\Big)$$

$$= xy\frac{\partial^2 f}{\partial z\partial x} - x^2\frac{\partial^2 f}{\partial z\partial y} - yz\frac{\partial^2 f}{\partial x^2} + z\frac{\partial f}{\partial y} + xz\frac{\partial^2 f}{\partial x\partial y}$$

$$-y\frac{\partial f}{\partial z} - xy\frac{\partial^2 f}{\partial x\partial z} + yz\frac{\partial^2 f}{\partial x^2} + x^2\frac{\partial^2 f}{\partial y\partial z} - xz\frac{\partial^2 f}{\partial y\partial x}$$

$$= z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}$$

$$= A_1 f\,.$$

Thus, we can summarize these results as

$$[A_i, A_j] = \varepsilon_{ijk} A_k\,.$$

Similarly, for the B_i, we calculate the pertinent commutators as

$$[B_1, B_2]f = B_1(B_2 f) - B_2(B_1 f)$$

$$= \left(x\frac{\partial}{\partial w} - w\frac{\partial}{\partial x}\right)\left(y\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial y}\right) - \left(y\frac{\partial}{\partial w} - w\frac{\partial}{\partial y}\right)\left(x\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial x}\right)$$

$$= xy\frac{\partial^2 f}{\partial w^2} - x\frac{\partial f}{\partial y} - xw\frac{\partial^2 f}{\partial w \partial y} - yw\frac{\partial^2 f}{\partial x \partial w} + w^2\frac{\partial^2 f}{\partial x \partial y}$$

$$-xy\frac{\partial^2 f}{\partial w^2} + y\frac{\partial f}{\partial x} + yw\frac{\partial^2 f}{\partial w \partial x} + xw\frac{\partial^2 f}{\partial y \partial t} - w^2\frac{\partial^2 f}{\partial y \partial x}$$

$$= y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}$$

$$= A_3 f\,,$$

$$[B_1, B_3]f = B_1(B_3 f) - B_3(B_1 f)$$

$$= \left(x\frac{\partial}{\partial w} - w\frac{\partial}{\partial x}\right)\left(z\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial z}\right) - \left(z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}\right)\left(x\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial x}\right)$$

$$= xz\frac{\partial^2 f}{\partial w^2} - x\frac{\partial f}{\partial z} - xw\frac{\partial^2 f}{\partial w \partial z} - zw\frac{\partial^2 f}{\partial x \partial w} + t^2\frac{\partial^2 f}{\partial x \partial z}$$

$$-xz\frac{\partial^2 f}{\partial w^2} + z\frac{\partial f}{\partial x} + zw\frac{\partial^2 f}{\partial w \partial x} + xt\frac{\partial^2 f}{\partial z \partial w} - w^2\frac{\partial^2 f}{\partial z \partial x}$$

$$= z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}$$

$$= -A_2 f\,,$$

$$[B_2, B_3]f = B_2(B_3 f) - B_3(B_2 f)$$

$$= \left(y\frac{\partial}{\partial w} - w\frac{\partial}{\partial y}\right)\left(z\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial z}\right) - \left(z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}\right)\left(y\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial y}\right)$$

$$= yz\frac{\partial^2 f}{\partial w^2} - y\frac{\partial f}{\partial z} - yw\frac{\partial^2 f}{\partial w \partial z} - zw\frac{\partial^2 f}{\partial y \partial w} + w^2\frac{\partial^2 f}{\partial y \partial z}$$

$$-yz\frac{\partial^2 f}{\partial w^2} + z\frac{\partial f}{\partial y} + zw\frac{\partial^2 f}{\partial w \partial y} + yt\frac{\partial^2 f}{\partial z \partial w} - w^2\frac{\partial^2 f}{\partial z \partial y}$$

$$= z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}$$

$$= A_1 f\,.$$

These results can be summarized as

$$[B_i, B_j] = \varepsilon_{ijk} A_k\,.$$

Finally, for the commutators between the A_i and B_j, we note first by inspection that

$$[A_i, B_i] = 0\,,$$

for $i = 1, 2, 3$, since A_i and B_i involve mutually exclusive pairs of variables. For the remaining commutator pairs, we have

$$[A_1, B_2] = A_1(B_2 f) - B_2(A_1 f)$$

$$= \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\left(y\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial y}\right) - \left(y\frac{\partial}{\partial w} - w\frac{\partial}{\partial y}\right)\left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right)$$

$$= z\frac{\partial f}{\partial w} + yz\frac{\partial^2 f}{\partial y \partial w} - zw\frac{\partial^2 f}{\partial y^2} - y^2\frac{\partial^2 f}{\partial z \partial w} + yw\frac{\partial^2 f}{\partial z \partial y}$$

$$-yz\frac{\partial^2 f}{\partial w \partial y} + y^2\frac{\partial^2 f}{\partial w \partial z} + zw\frac{\partial^2 f}{\partial y^2} - w\frac{\partial f}{\partial z} - yw\frac{\partial^2 f}{\partial y \partial z}$$

$$= z\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial z}$$

$$= B_3 f\,,$$

$$[A_1, B_3] = A_1(B_3 f) - B_3(A_1 f)$$

$$= \left(z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\left(z\frac{\partial f}{\partial w} - w\frac{\partial f}{\partial z}\right) - \left(z\frac{\partial}{\partial w} - w\frac{\partial}{\partial z}\right)\left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right)$$

$$= z^2 \frac{\partial^2 f}{\partial y \partial w} - zw \frac{\partial^2 f}{\partial y \partial z} - y \frac{\partial f}{\partial w} - yz \frac{\partial^2 f}{\partial z \partial w} + yw \frac{\partial^2}{\partial z^2}$$

$$-z^2 \frac{\partial^2 f}{\partial w \partial y} + yz \frac{\partial^2 f}{\partial w \partial z} + w \frac{\partial f}{\partial y} + zw \frac{\partial^2 f}{\partial z \partial y} - yw \frac{\partial^2 f}{\partial z^2}$$

$$= w \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial w}$$

$$= -B_2 f \, ,$$

$$[A_2, B_3] = A_2(B_3 f) - B_3(A_2 f)$$

$$= \left(x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x} \right) \left(z \frac{\partial f}{\partial w} - w \frac{\partial f}{\partial z} \right) - \left(z \frac{\partial}{\partial w} - w \frac{\partial}{\partial z} \right) \left(x \frac{\partial f}{\partial z} - z \frac{\partial f}{\partial x} \right)$$

$$= x \frac{\partial f}{\partial w} + xz \frac{\partial^2 f}{\partial z \partial w} - xw \frac{\partial^2 f}{\partial z^2} - z^2 \frac{\partial^2 f}{\partial x \partial w} + zw \frac{\partial^2 f}{\partial x \partial z}$$

$$-xz \frac{\partial^2 f}{\partial w \partial z} + z^2 \frac{\partial^2 f}{\partial w \partial x} - w \frac{\partial f}{\partial x} + xw \frac{\partial^2 f}{\partial z^2} - zw \frac{\partial^2 f}{\partial x \partial z}$$

$$= x \frac{\partial f}{\partial w} - w \frac{\partial f}{\partial x}$$

$$= B_1 f \, .$$

Thus,

$$[A_i, B_j] = \varepsilon_{ijk} B_k \, .$$

9. Consider the following linear combinations of the operators in Problem 7:

$$J_i = \tfrac{1}{2}(A_i + B_i), \qquad K_i = \tfrac{1}{2}(A_i - B_i) \, . \tag{3}$$

We can now use the commutation relations derived in Problem 8 to derive the commutation relations for the J_i and K_j. For the J_i, we have

$$
\begin{aligned}
[J_i, J_j] &= \tfrac{1}{4}[A_i + B_i, A_j + B_j] \\
&= \tfrac{1}{4}\Big([A_i, A_j] + [A_i, B_j] + [B_i, A_j] + [B_i, B_j]\Big) \\
&= \tfrac{1}{4}(\varepsilon_{ijk}A_k + \varepsilon_{ijk}B_k + \varepsilon_{ijk}B_k + \varepsilon_{ijk}A_k) \\
&= \varepsilon_{ijk}\tfrac{1}{2}(A_k + B_k) \\
&= \varepsilon_{ijk}J_k\,,
\end{aligned}
$$

$$
\begin{aligned}
[K_i, K_j] &= \tfrac{1}{4}[A_i - B_i, A_j - B_j] \\
&= \tfrac{1}{4}\Big([A_i, A_j] - [A_i, B_j] - [B_i, A_j] + [B_i, B_j]\Big) \\
&= \tfrac{1}{4}(\varepsilon_{ijk}A_k - \varepsilon_{ijk}B_k - \varepsilon_{ijk}B_k + \varepsilon_{ijk}A_k) \\
&= \varepsilon_{ijk}\tfrac{1}{2}(A_k - B_k) \\
&= \varepsilon_{ijk}K_k\,,
\end{aligned}
$$

$$
\begin{aligned}
[J_i, K_j] &= \tfrac{1}{4}[A_i + B_i, A_j - B_j] \\
&= \tfrac{1}{4}\Big([A_i, A_j] - [A_i, B_j] + [B_i, A_j] - [B_i, B_j]\Big) \\
&= \tfrac{1}{4}(\varepsilon_{ijk}A_k + \varepsilon_{ijk}B_k - \varepsilon_{ijk}B_k - \varepsilon_{ijk}A_k) \\
&= 0\,.
\end{aligned}
$$