Bogoliubov Theory (bosons)

We have a Hamiltonian of the form

$$H_b = \psi^+ D\,\psi \qquad D = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^* & \mathbf{A}^* \end{pmatrix} \qquad \psi = \begin{pmatrix} \mathbf{a} \\ \mathbf{b}^+ \end{pmatrix} \tag{1}$$

where $\mathbf{a} = (a_1, \ldots, a_n)$ and similarly for $\mathbf{b}$. For H_b to be Hermitian, we need $D^+ = D$ ($\mathbf{A} = \mathbf{A}^+$ and $\mathbf{B} = \mathbf{B}^{\mathrm{T}}$).

The set of n states associated with $\mathbf{a}$ are related to $\mathbf{b}$ by a global symmetry, such as time reversal in the application to superfluidity. We require that the Hamiltonian is invariant under this global symmetry transformation. This means:

$$D^* = \Sigma_x D \Sigma_x \qquad \Sigma_x = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \tag{2}$$

Transformation

We want a transformation $\psi \to \phi = T\psi$. Since we must have $H_b = \psi^+ D\psi = \phi^+ \tilde{D}\phi$, D transforms as $D \to \tilde{D} = (T^{-1})^+ D\,T^{-1}$. The transformation should preserve symmetries of the Hamiltonian, *i.e.* $\tilde{D}$ should obey the symmetry (2). This implies that $(T^{-1})^* = \Sigma_x T^{-1} \Sigma_x$. Equivalently, T^{-1} should be of the form:

$$T^{-1} = \begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V}^* & \mathbf{U}^* \end{pmatrix} \tag{3}$$

The transformation should also preserve commutation relations. Let us write $\phi = (\boldsymbol{\alpha}, \boldsymbol{\beta}^+)$ and impose bosonic commutation relations for α_i and β_i. Then, the commutators $[a_i, a_j^+] = \delta_{ij}$ and $[a_i, b_j] = 0$ give us $\mathbf{U}\mathbf{U}^+ - \mathbf{V}\mathbf{V}^+ = \mathbf{1}$ and $\mathbf{U}\mathbf{V}^T - \mathbf{V}\mathbf{U}^T = \mathbf{0}$. More concisely, $T^{-1}\Sigma_z (T^{-1})^+ = \Sigma_z$ where $\Sigma_z = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$. Equivalently,

$$T \Sigma_z T^+ = \Sigma_z \tag{4}$$

(This makes sense since both T and T^{-1} are Bogoliubov transforms.) This defines the group of Bogoliubov transformations. We can now invert T^{-1}:

$$T = \Sigma_z (T^{-1})^+ \Sigma_z = \begin{pmatrix} \mathbf{U}^+ & -\mathbf{V}^T \\ -\mathbf{V}^+ & \mathbf{U}^T \end{pmatrix}$$

Notice that Σ_z is invariant under T:

$$\Sigma_z \to \tilde{\Sigma}_z = (T^{-1})^+ \Sigma_z T^{-1} = (\Sigma_z T \Sigma_z)\Sigma_z T^{-1} = \Sigma_z$$

So, we see that the norm $I = \psi^+ \Sigma_z \psi = \phi^+ \tilde{\Sigma}_z \phi$ is conserved. Physically, this is a consequence of that fact that H_b only creates and annihilates time-reversed states in pairs, *i.e.* terms of the form $a^+ b^+$ but not $a^+ a^+$ or $b^+ b^+$.

Diagonalization

We also want to diagonalize H_b. In other words, we want to find a transformation T such that $H_b = \psi^+ D\psi = \phi^+ \Lambda \phi$ where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots)$ contains the eigenvalues of H_b, *i.e.*

$$(T^{-1})^+ D\,T^{-1} \;=\; \Lambda \tag{5}$$

$$\Rightarrow D\,T^{-1} \;=\; T^+ \Lambda = (\Sigma_z T^{-1} \Sigma_z)\,\Lambda \tag{6}$$

From each column of the last equation, we obtain the secular equation

$$(D - \lambda_i \Sigma_z) t_i = 0 \tag{7}$$

where t_i is a column of T^{-1}. From the symmetry (2), we can see that if $t_i = (\mathbf{u}, \mathbf{v}^*)$ is an eigenvector with eigenvalue λ_i, then $(\mathbf{v}, \mathbf{u}^*)$ is also an eigenvector with eigenvalue -λ_i. This is explicit in the form of T^{-1} in (3). So, the entries in the diagonalized matrix Λ have the relation $\lambda_{i+n} = \lambda_i$.

Connection to fermions

For each H_b, there is a fermionic Bogoliubov-de-Gennes Hamiltonian, $H_f = \chi^+ G \chi$ ($G^+ = G$), with the same spectrum. In the fermionic problem, we require the diagonalization:

$$U^+ G U = \Lambda' \tag{8}$$

for some unitary transformation U ($\chi \to U^+ \chi$) and diagonal Λ'.

H_b should have a non-negative spectrum: $\lambda_i \geq 0$. This means that D can be decomposed as

$$D = K^+ K \tag{9}$$

We can choose K to obey the same symmetry (2) as D:

$$K^* = \Sigma_x K \Sigma_x. \tag{10}$$

Substituting this decomposition into (5), we get

$$(K T^{-1})^+ K T^{-1} = \Lambda \tag{11}$$

so we see that $U = K T^{-1} \Lambda^{-1/2}$ is a unitary matrix. We can re-express the symmetry property (4) of T in terms of U:

$$(\Lambda^{-1/2} U^+ K) \Sigma_z (K^+ U \Lambda^{-1/2}) \;=\; \Sigma_z$$

$$\Rightarrow U^+ (K \Sigma_z K^+) U \;=\; \Sigma_z \Lambda \tag{12}$$

Comparing with (8), we see that a fermionic Hamiltonian H_f will have eigenvalues $\pm\lambda_i$ if we identify:

$$G = K^+ \Sigma_z K \tag{13}$$

We see that H_f and H_b share the same eigenvalues (but quite different eigenvectors).

From the symmetry (10) of K, we check that G has the correct symmetry for the fermionic problem:

$$\Sigma_x\, G\, \Sigma_x = -G^* \qquad G = \begin{pmatrix} \mathbf{h} & \boldsymbol{\Delta} \\ -\boldsymbol{\Delta}^* & -\mathbf{h}^{\mathrm{T}} \end{pmatrix} \qquad \begin{matrix} \mathbf{h}^+ = \mathbf{h} \\ \boldsymbol{\Delta}^{\mathrm{T}} = -\boldsymbol{\Delta} \end{matrix} \tag{14}$$