NOTES ON SPIN WAVES

Consider a small rotation $\mathsf{R}(\mathbf{p})$ of the spins around $\mathbf{p}$, so that the spins rotate by $\mathbf{S} \to \mathsf{R}\mathbf{S}$:

$$\mathsf{R} = e^{\mathsf{P}}; \qquad [\mathsf{P}]^{ab} = \epsilon^{abc} p^c \tag{1}$$

So, $\delta\mathbf{S} \simeq -\mathbf{p} \times \mathbf{S} + \mathbf{p} \times (\mathbf{p} \times \mathbf{S})/2$.
This parametrisation is related to Ginzburg (1978) by $\mathbf{p} = 2\varphi$.
The **charge density**, given by: $\rho = \epsilon_{ij}\mathbf{S} \cdot (\partial_i \mathbf{S} \times \partial_j \mathbf{S})/2$, becomes:

$$\rho + \delta\rho = \frac{1}{2}\epsilon_{ij}\mathsf{R}\mathbf{S} \cdot [\partial_i(\mathsf{R}\mathbf{S}) \times \partial_j(\mathsf{R}\mathbf{S})]$$

so that

$$\begin{aligned}\delta\rho &= \frac{1}{2}\epsilon_{ij}\mathbf{S} \cdot \Big[2(\mathsf{R}^{-1}\partial_i\mathsf{R})\mathbf{S} \times \partial_j\mathbf{S} + (\mathsf{R}^{-1}\partial_i\mathsf{R})\mathbf{S} \times (\mathsf{R}^{-1}\partial_j\mathsf{R})\mathbf{S}\Big] \\ &= -\epsilon_{ij}\partial_i\mathbf{p} \cdot \partial_j\mathbf{S} + \frac{1}{2}\epsilon_{ij}\partial_i\mathbf{p} \cdot \partial_j(\mathbf{p} \times \mathbf{S}) + O(p^3)\end{aligned} \tag{2}$$

using $(\mathsf{R}^{-1}\partial_i\mathsf{R})\mathbf{S} \simeq (\partial_i\mathsf{P} - [\mathsf{P}, \partial_i\mathsf{P}]/2)\mathbf{S} = \mathbf{S} \times \partial_i\mathbf{p} + (\partial_i\mathbf{p} \times \mathbf{p}) \times \mathbf{S}/2$. We can drop higher-order terms for $\delta\rho$ if $|\mathbf{p}| \ll 1$ and $|\nabla\mathbf{p}| \ll |\nabla\mathbf{S}|$ (so that $\delta\rho/\rho \ll 1$.)

To first order in p, the component of $\mathbf{p}$ parallel to $\mathbf{S}$ does not affect the charge density, so we can choose $\mathbf{p} \cdot \mathbf{S} = 0$. At this order, $\delta\rho$ can be written as:

$$\delta\rho = \partial_i\mathcal{E}_i\ , \qquad \mathcal{E}_i = \epsilon_{ij}(-\mathbf{p} \cdot \partial_j\mathbf{S} + \partial_j a) \tag{3}$$

for some "gauge field" a. From continuity $\delta\dot{\rho} + \partial_i J_i = 0$, the **current density** is:

$$J_i = \epsilon_{ij}\dot{\mathbf{p}} \cdot \partial_j\mathbf{S} \quad (+\text{ divergence-free part}) \tag{4}$$

Note that we have only identified the *transport current*, and not any circulating current in the bulk. Therefore, this would not give the correct σ_{xy}.♠*Need Faraday's law for motion of CS flux?*♠

For each disorder realisation, the $T = 0$ current correlation function is related to the spin dynamics by:

$$\begin{aligned}\langle J_i(\mathbf{r},t)J_j(\mathbf{r}',t')\rangle &= \Big[\delta_{ij}\partial_k S^a(\mathbf{r})\partial_k S^b(\mathbf{r}') - \partial_j S^a(\mathbf{r})\partial_i S^b(\mathbf{r}')\Big]\langle\dot{p}^a(\mathbf{r},t)\dot{p}^b(\mathbf{r}',t')\rangle \\ &= S^a(\mathbf{r})S^b(\mathbf{r}')\Big[\delta_{ij}\langle\partial_k\dot{p}^a(\mathbf{r},t)\partial_k\dot{p}^b(\mathbf{r}',t')\rangle - \langle\partial_j\dot{p}^a(\mathbf{r},t)\partial_i\dot{p}^b(\mathbf{r}',t')\rangle\Big]\end{aligned} \tag{5}$$

The second form is obtained using $J_i = -\epsilon_{ij}\mathbf{S} \cdot \partial_j\dot{\mathbf{p}}$. The off-diagonal part is symmetrical and should average to zero (ignoring UCF). The trace is given by:

$$\begin{aligned}\langle J_i(\mathbf{r},t)J_i(\mathbf{r}',t')\rangle &= \partial_k S^a(\mathbf{r})\partial_k S^b(\mathbf{r}')\langle\dot{p}^a(\mathbf{r},t)\dot{p}^b(\mathbf{r}',t')\rangle \\ &= S^a(\mathbf{r})S^b(\mathbf{r}')\langle\partial_k\dot{p}^a(\mathbf{r},t)\partial_k\dot{p}^b(\mathbf{r}',t')\rangle\end{aligned} \tag{6}$$

To second order in p, we can write $\delta\rho$ in terms of the components of $\mathbf{p}$ normal and parallel to $\mathbf{S}$: $\mathbf{p} = \mathbf{p}_\perp + p_{||}\mathbf{S}$. Then, $\delta\rho = \delta\rho_\perp + \delta\rho'$ where:

$$\delta\rho_\perp = \partial_i \mathcal{E}_i + \frac{1}{2}\epsilon_{ij}\left\{(\partial_i \mathbf{p}_\perp \times \partial_j \mathbf{p}_\perp)\cdot \mathbf{S} + \partial_i \mathbf{p}_\perp \cdot (\mathbf{p}_\perp \times \partial_j \mathbf{S})\right\} \tag{7}$$

$$\delta\rho' = \frac{1}{2}\epsilon_{ij}\partial_i(p_{||}\mathbf{p}_\perp)\cdot(\partial_j \mathbf{S}\times\mathbf{S}) \tag{8}$$

Note that the linear-p term does not depend on $p_{||}$. Also, $\delta\rho'$ vanishes if we pick $\mathbf{p}\cdot\mathbf{S} = 0$. (In fact, it is a divergence: $\partial_i(p_{||}p_{i\perp}\rho)/2$, if we write $\mathbf{p}_\perp = p_{i\perp}\partial_i\mathbf{S}$.)

Exchange energy

$$H_J = \frac{1}{2}|\nabla\mathbf{S}|^2 \to H + \delta H_J = \frac{1}{2}|\nabla\mathbf{S} + (\mathsf{R}^{-1}\nabla\mathsf{R})\mathbf{S}|^2 \tag{9}$$

$$\begin{aligned}\Rightarrow \delta H_J &= (\mathsf{R}^{-1}\partial_i\mathsf{R})\mathbf{S}\cdot\partial_i\mathbf{S} + \frac{1}{2}|(\mathsf{R}^{-1}\partial_i\mathsf{R})\mathbf{S}|^2 \\ &= \partial_i\mathbf{p}\cdot(\partial_i\mathbf{S}\times\mathbf{S}) \quad + \\ &\quad \frac{1}{2}|\nabla\mathbf{p}|^2 - (\nabla\mathbf{p}\cdot\mathbf{S})^2 + \frac{1}{2}(\partial_i\mathbf{p}\cdot\mathbf{S})\partial_i(\mathbf{p}\cdot\mathbf{S}) - \frac{1}{2}(\mathbf{p}\cdot\mathbf{S})(\partial_i\mathbf{p}\cdot\partial_i\mathbf{S})\end{aligned} \tag{10}$$

This can be separated into a term involving the component of $\mathbf{p}$ normal to $\mathbf{S}$ and other terms. Writing $\mathbf{p} = \mathbf{p}_\perp + p_{||}\mathbf{S}$, the terms of $O(p^2)$ are $\delta H_J = \delta H_{J\perp} + \delta H'_J$ with:

$$\delta H_{J\perp} = \partial_i\mathbf{p}_\perp\cdot(\partial_i\mathbf{S}\times\mathbf{S}) + \frac{1}{2}|\nabla\mathbf{p}_\perp|^2 - |\nabla\mathbf{p}_\perp\cdot\mathbf{S}|^2 \tag{11}$$

$$\delta H'_J = \frac{1}{2}\partial_i(p_{||}\mathbf{p}_\perp)\cdot\partial_i\mathbf{S} \tag{12}$$

The exchange energy does depend on $p_{||}$ at order $O(p^2)$. If we integrate by parts, then $\delta H'_J$ vanishes in the regions where $\partial^2\mathbf{S}$ and $\mathbf{S}$ are collinear.

Potential energy

$$\begin{aligned}\delta H_\rho &= V_{\rm H}(\mathbf{r})\delta\rho(\mathbf{r}) + \frac{1}{2}\int \delta\rho(\mathbf{r})U(\mathbf{r}-\mathbf{r}')\delta\rho(\mathbf{r}')d^2\mathbf{r}' \\ V_{\rm H}(\mathbf{r}) &= V_{\rm bare}(\mathbf{r}) + \int U(\mathbf{r}-\mathbf{r}')\rho(\mathbf{r}')d^2\mathbf{r}'.\end{aligned} \tag{13}$$

where $V_{\rm H}$ is the Hartree potential, $V_{\rm bare}$ is the bare potential measured from the chemical potential μ, and U is the interaction. (The unit of energy is the exchange coupling.)

Note that there are two contributions to the energy density involving $p_{||}$: $V_{\rm H}(\mathbf{r})\delta\rho'(\mathbf{r})$ and $H'_J(\mathbf{r})$. These cancel (after integrating over all space) if $\mathbf{S}$ is the ground-state configuration (17).

Ground state and Fluctuations

In the ground state, $\delta H_J + \delta H_\rho$ should not have a linear-p term:

$$\begin{aligned} &\int \partial_i \mathbf{p} \cdot \{\partial_i \mathbf{S} \times \mathbf{S} - \epsilon_{ij} V_{\mathrm{H}}(\mathbf{r}) \partial_j \mathbf{S}\}\ d^2\mathbf{r} = 0 \quad \forall \mathbf{p} \\ \Rightarrow\ &\partial_i \left[\partial_i \mathbf{S} \times \mathbf{S} - \epsilon_{ij} V_{\mathrm{H}}(\mathbf{r}) \partial_j \mathbf{S}\right] = 0 \end{aligned} \tag{14}$$

This means that

$$\epsilon_{ij} \partial_j \mathbf{S} \times \mathbf{S} + V_{\mathrm{H}} \partial_i \mathbf{S} = \partial_i \mathbf{C} \tag{15}$$

for some $\mathbf{C}(\mathbf{r})$. Belavin-Polyakov gives $\mathbf{C} = \pm \mathbf{S}$ for $V_{\mathrm{H}} = 0$, but I cannot deduce this from here.
It also means that:

$$\partial^2 \mathbf{S} \times \mathbf{S} = \epsilon_{ij} (\partial_i V_{\mathrm{H}}) \partial_j \mathbf{S} \tag{16}$$

$$\Rightarrow \partial^2 \mathbf{S} + \epsilon_{ij} (\partial_i V_{\mathrm{H}}) \partial_j \mathbf{S} \times \mathbf{S} = -|\nabla \mathbf{S}|^2 \mathbf{S} \tag{17}$$

(I have used $\partial_i (\mathbf{S} \cdot \partial_i \mathbf{S}) = \mathbf{S} \cdot \partial^2 \mathbf{S} + \partial_i \mathbf{S} \cdot \partial_i \mathbf{S} = 0$.) This is the condition for a stationary configuration that the spin should be collinear with a local effective field which has contributions from the exchange term and the Hartree term. (See eqn (28).) This condition does not depend on μ, because we are only considering variations at fixed global charge.

A Belavin-Polyakov skyrmion of charge N can be written in the form:

$$\tan(\theta/2) = (R/r)^{|N|} \qquad \phi = N\varphi \tag{18}$$

where θ and ϕ are the polar angles in spin space and φ is the azimuthal angle around the origin in real space.

We can use Polyakov's parametrization of the spin configuration:

$$\begin{aligned} \partial_i \mathbf{S} &= B_i^a \mathbf{e}_a \\ \partial_i \mathbf{e}_a &= A_i^{ab} \mathbf{e}_b - B_i^a \mathbf{S} \end{aligned} \tag{19}$$

where we choose the local orthonormal basis in spin space to be ($\mathbf{e}_1$,$\mathbf{e}_2$,$\mathbf{S}$) such that $\mathbf{e}_a \times \mathbf{e}_b = \epsilon_{ab} \mathbf{S}$. Orthonormality requires that $A_i^{ab} = -A_i^{ba} = \epsilon_{ab} A_i$. Note that:

$$\epsilon_{ij} \partial_i \partial_j \mathbf{S} = 0 \quad \Rightarrow \quad \epsilon_{ij} D_i^{ab} B_j^b = 0 \qquad\qquad D_i^{ab} \equiv \delta_{ab} \partial_i - A_i^{ab} \tag{20}$$

$$\epsilon_{ij} \partial_i \partial_j \mathbf{e}_a = 0 \quad \Rightarrow \quad \epsilon_{ij} (D_i^{bc} A_j^{ac} + B_i^a B_j^b) = 0 \tag{21}$$

The topological charge density (quantized in units of 2π) can be written in terms of $F_{ij} \equiv \partial_i A_j - \partial_j A_i$:

$$\epsilon_{ij} \mathbf{S} \cdot (\partial_i \mathbf{S} \times \partial_j \mathbf{S})/4 = (B_1^1 B_2^2 - B_1^2 B_2^1)/2 = F_{21}/2 \tag{22}$$

It is also convenient to use the complex notation: $b_i = B_i^1 + iB_i^2$, and $D_i \equiv \partial_i + iA_i$. If we pick $\mathbf{e}_2$ to be normal to the z-direction in spin space, then $b_i = (\mathbf{e}_1 + i\mathbf{e}_2)\cdot\partial_i\mathbf{S} = \partial_i\theta + i\sin\theta\,\partial_i\phi$ in polar coordinates.

Then, eqns. (20-21) become:

$$\epsilon_{ij} D_i b_j = 0, \qquad F_{ij} = i(b_i^* b_j - b_j^* b_i)/2 \tag{23}$$

Equation (15) for the ground state configuration now becomes:

$$-iD_i b_i - \epsilon_{ij}(\partial_i V_{\mathrm{H}})b_j = 0 \tag{24}$$

Note that the choice of basis corresponds to a gauge transformation. A rotation of the basis vectors through an angle χ implies:

$$A_i \to A_i + \partial_i\chi \qquad b_i \to b_i e^{-i\chi} \tag{25}$$

The Belavin-Polyakov self-dual solutions have the additional property:

$$\partial_i\mathbf{S} = \pm\epsilon_{ij}\mathbf{S}\times\partial_j\mathbf{S} \quad \Rightarrow \quad B_i^a = \mp\epsilon_{ij}\epsilon_{ab}B_j^b \tag{26}$$

$$\Rightarrow \quad B_i^1 B_i^2 = 0, \quad B_i^1 B_i^1 = B_i^2 B_i^2, \quad b_i b_i = 0, \quad F_{21} = \mp b_i^* b_i/2 \tag{27}$$

Equation of motion

The semiclassical equation of motion is given by:

$$\dot{\mathbf{S}} = -\mathbf{S}\times\frac{\delta H}{\delta\mathbf{S}} = \mathbf{S}\times\mathbf{H}_{\text{eff}}(\mathbf{S})$$

$$\text{where} \quad \mathbf{H}_{\text{eff}} \equiv \partial^2\mathbf{S} + \epsilon_{ij}\left(\partial_i V_{\mathrm{H}}[\mathbf{S}]\right)\partial_j\mathbf{S}\times\mathbf{S} \tag{28}$$

is the same as that on the lhs of eqn (17). Alternatively, we can use $\dot{\mathbf{S}} = i[H, \mathbf{S}]$ and the commutators:

$$\left[S^a(\mathbf{r}), S^b(\mathbf{r}')\right] = i\epsilon^{abc}S^c(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\ , \qquad \left[\partial_i S^a(\mathbf{r}), S^b(\mathbf{r}')\right] = i\epsilon^{abc}S^c(\mathbf{r}')\partial_i\delta(\mathbf{r}-\mathbf{r}') \tag{29}$$

to obtain the quantum equation of motion.

From the equation of motion, we can see that the divergence of the **spin current** is: $\partial_i\mathbf{J}_i = \partial_i(\partial_i\mathbf{S}\times\mathbf{S}) + \epsilon_{ij}\partial_i V_{\mathrm{H}}\partial_j\mathbf{S}$.

Linearizing eqn (28) and substituting for $\partial^2\mathbf{S}$ using (17) gives:

$$\dot{\mathbf{p}}\times\mathbf{S} = (\mathbf{p}\times\mathbf{S})\times\partial^2\mathbf{S} + \mathbf{S}\times\partial^2(\mathbf{p}\times\mathbf{S}) + \epsilon_{ij}\partial_i V_{\mathrm{H}}\partial_j(\mathbf{p}\times\mathbf{S}) - \epsilon_{ij}\partial_i(\delta V_{\mathrm{H}}[\mathbf{p}])\partial_j\mathbf{S}$$

$$\text{or} \quad \dot{\mathbf{p}}_\perp = \mathbf{S}\times\partial^2\mathbf{p}_\perp - 2(\mathbf{S}\cdot\partial_i\mathbf{p}_\perp)\,\mathbf{S}\times\partial_i\mathbf{S} + \epsilon_{ij}\partial_i V_{\mathrm{H}}[\partial_j\mathbf{p}_\perp]_\perp - \epsilon_{ij}\partial_i(\delta V_{\mathrm{H}}[\mathbf{p}])\,\mathbf{S}\times\partial_j\mathbf{S} \tag{30}$$

where $[\ldots]_\perp$ projects out the components parallel to $\mathbf{S}$, and $\delta V_{\mathrm{H}}[\mathbf{p}]$ is the change in V_{H} due to $\mathbf{p}$. It is in general a non-local function of $\mathbf{p}$:

$$\delta V_{\rm H}[\mathbf{p}] = \int U(\mathbf{r}-\mathbf{r}')\,\delta\rho(\mathbf{r}')\,d\mathbf{r}' = (\epsilon_{ij}/2)\int U(\mathbf{r}-\mathbf{r}')\,\partial_i \mathbf{S}(\mathbf{r}')\cdot\partial_j \mathbf{p}(\mathbf{r}')\,d\mathbf{r}' \tag{31}$$

A global rotation corresponds to a constant $\mathbf{p}$, but not a constant $\mathbf{p}_\perp$. A constant $\mathbf{p}_\perp$ appears to satisfy eqn (30) because the difference is a function of $p_{||}$ which does not enter into the equation of motion at $O(p)$. However, a constant $\mathbf{p}_\perp$ is not in general possible for a non-collinear spin configuration.

Let us write $\mathbf{p}_\perp = p_a \mathbf{e}_a$ and $\psi = p_1 + ip_2$. Then, the density fluctuations are:

$$\delta\rho = \epsilon_{ij}\,\mathrm{Re}[b_i^*\,(D_j\psi)] \tag{32}$$

It can be shown that the equation of motion becomes:

$$i\dot{\psi} = -D_i D_i \psi + i\epsilon_{ij}(\partial_i V_{\rm H})D_j\psi - (b_i^* b_i \psi + b_i b_i \psi^*)/2 - \epsilon_{ij}\partial_i(\delta V_{\rm H})b_j \tag{33}$$

The choice of spin basis corresponds to a gauge transformation (25). The wavefunction transforms as $\psi \to \psi e^{-i\chi}$, and $D_i\psi$ is its covariant derivative. And so, $b_i^*\psi$ and $b_i^* D_j\psi$ are gauge-invariant objects.

There are 3 zero modes, corresponding to the three possible constant $\mathbf{p}$:

$$D_i\psi = -p_{||}b_i, \qquad (\psi^* b_i + \psi b_i^*)/2 = \partial_i p_{||}, \qquad p_{||} = \mathbf{p}\cdot\mathbf{S} \tag{34}$$

The corresponding Hamiltonian density for the field ψ is:

$$H = |(-i\partial_i + \tilde{A}_i)\psi|^2 - \frac{1}{4}|\nabla V_{\rm H}|^2|\psi|^2 - \frac{1}{4}\Psi^+(\mathbf{r})M_{ii}(\mathbf{r},\mathbf{r})\Psi(\mathbf{r}) + (\delta V_{\rm H}\text{ term}) \tag{35}$$

where $\Psi = (\psi, \psi^*)$, and the "magnon" sees a gauge field $\tilde{A}_i = A_i + \epsilon_{ij}\partial_j V_{\rm H}/2$, corresponding to a magnetic field of $F_{21} + \nabla^2 V_{\rm H}/2$. ♠*Physical explanation of extra term?*♠ The matrix $M(\mathbf{r},\mathbf{r}')$ is:

$$M_{ij}(\mathbf{r},\mathbf{r}') = \begin{pmatrix} b_i(\mathbf{r})b_j^*(\mathbf{r}') & b_i(\mathbf{r})b_j(\mathbf{r}') \\ b_i^*(\mathbf{r})b_j^*(\mathbf{r}') & b_i^*(\mathbf{r})b_j(\mathbf{r}') \end{pmatrix} \tag{36}$$

Note that the original particle current is given by $J_i = \epsilon_{ij}(\dot{\psi}^* b_j + \dot{\psi} b_j^*)/2$ so that the current correlation function is:

$$\langle J_i(\mathbf{r},t)J_i(\mathbf{r}',t')\rangle = \frac{1}{4}\langle\dot{\Psi}^+(\mathbf{r},t)M_{ii}(\mathbf{r},\mathbf{r}')\dot{\Psi}(\mathbf{r}',t')\rangle \tag{37}$$

John's Parametrization

Let us write $\delta\mathbf{S} = q_i\partial_i\mathbf{S}$. This is valid except along lines of zero density. Then

$$\mathbf{p} = q_i\,\partial_i\mathbf{S}\times\mathbf{S}\ , \qquad \delta\rho = \partial_i(\rho q_i)\ , \qquad J_i = -\rho\dot{q}_i \tag{38}$$

We can separate q_i into a gradient and a curl:

$$\rho q_i = \partial_i\phi + \epsilon_{ij}\partial_j\chi \quad , \qquad \delta\rho = \partial^2\phi \tag{39}$$

Note that the density change affects only curl-free part of q_i. The divergence-free contribution from χ does not affect the charge.

In terms of Polyakov's notation, $\mathbf{p} \cdot \partial_i \mathbf{S} = \rho \epsilon_{ij} q_j = (\psi b_i^* + \psi^* b_i)/2$. Since $\rho = i\epsilon_{ij} b_i^* b_j/2$, we see that: $\psi = i q_i b_i$. Also,

$$\frac{1}{4} \Psi^{+}(\mathbf{r}) M_{ij}(\mathbf{r}, \mathbf{r}') \Psi(\mathbf{r}') = \rho(\mathbf{r}) \rho(\mathbf{r}') \left[q_k(\mathbf{r}) q_k(\mathbf{r}') \delta_{ij} - q_i(\mathbf{r}) q_j(\mathbf{r}') \right] \tag{40}$$