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Bosons in a random potential: condensation and screening in a dense limit

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Abstract. We demonstrate that an extended Bose condensate can be stable in a random potential for a suitable weak-repulsive limit of a dense Bose gas, even though the non-interacting case is pathological. The condensate exists primarily because the interactions allow screening of the random potential. This may happen even when the chemical potential is in the Lifshitz tails of the single-particle case. Indeed, we argue that there are no Lifshitz tail states in our dense but weakly-interacting system. Using a number-phase representation, we calculate the increase in the depletion of the condensate with increasing randomness (at fixed density) which indicates the eventual destruction of the condensed phase—perhaps to a localized phase. The physical picture discussed should be relevant to the understanding of helium thin films.

1. Introduction

Helium absorbed in Vycor (Reppy 1984, Finotello *et al* 1988) or on various substrates (McQueeney *et al* 1984) have shown many properties that are not found in pure systems. Perhaps most striking is the absence of superfluidity at low concentrations. A natural interpretation of this phenomenon is that the ^{4}He atoms are localized at particular locations where there is a strong van der Waals attraction for them. However the mechanism for the onset of superfluidity, above a certain critical density, is less immediately obvious. It may be that there are charged anologues of this behaviour—granular superconductors and superconductors with short coherence lengths, but we will not consider the Coulombic case in this paper. (The effect of randomness on charged systems have been discussed, for example, by Azbel (1980) and Gold (1983).)

A common feature of the experiments is the heterogeneity of the substrates, albeit of a rather different nature in, say, Grafoil and Vycor. It is tempting to isolate the dominant aspect of the heterogeneity as randomness for a theoretical investigation. Indeed, it is widely believed that, at least in the ^{4}He systems, disorder is a key factor in the destruction of off-diagonal long range order (ODLRO) as the boson density is decreased. In this paper, we wish to discuss some of the theoretical ideas behind this hypothesis in the context of a Bose gas in two dimensions. Instead of looking directly at the disordered phase at low boson coverage, we will explore the robustness of the condensed phase in the limit of high density. In particular, we will emphasize the rôle of the Bose condensate in the partial *screening* of the random potential. This

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requires the bosons to be interacting (see below) and we discuss the case of short-range interactions in this paper. It will become clear that the dense Bose gas needs only weak repulsion to have a stable condensed phase.

Theoretically, the study of quantum motion in a random potential has been mainly concerned with the *single-particle* Schrödinger equation (see e.g. Lee and Ramakrishnan 1985). It is believed that, while the lowest eigenstates (Lifshitz tail states) are localized, there are extended states above a certain energy (mobility edge) for three dimensions and higher. In one and two dimensions, it is believed that all states are localized. This theory has been very successful for disordered electronic systems, given that it does not consider the effect of Coulombic interactions. Unfortunately, it is not immediately useful to Bose systems: non-interacting bosons will simply condense into the lowest eigenstate which is localized around a small region. To escape from this ridiculous situation, some repulsion between the bosons must be included so that the system has a finite compressibility (which is provided by the exclusion principle in Fermi systems). As already mentioned, we will focus on short-range repulsion in this paper. We will see that it is this finite compressibility that allows the bosons to screen out the random potential to some extent.

The first investigation of the problem of *interacting* bosons was concerned with the localized 'Bose glass' phase (Hertz *et al* 1979) but the condensed phase has since received more attention. Ma *et al* (1986) considered a lattice model with hard-core repulsion. The occupation number at each site is constrained to be 0 or 1. This can be mapped onto a spin-half ferromagnetic XY model with a random field in the z direction. It should be noted that disorder is an essential ingredient for this model—the ground state of the non-random XY model has ODLRO for all spins in all dimensions greater than one (Kennedy *et al* 1988). In a semiclassical picture, the spins will tend to line up their xy components at sites where the random field happens to be small. These sites polarize neighbouring spins and act as nucleation centres giving rise to ODLRO across the system. Ma *et al* pointed out that this XY ferromagnetism is destroyed at sufficiently strong disorder when the quantum nature of the spins is taken into account.

More recently, the destruction of ODLRO in the Bose problem was discussed in the context of a scaling hypothesis (Fisher and Fisher 1988). The general features of the phase diagram were discussed, partially supported by renormalisation group calculations (Fisher *et al* 1989, Weichman and Kim 1989). A *finite* on-site repulsion energy was used, as against a hard core, giving rise to the additional possibility of a Mott-insulating phase when the boson number is commensurate with the number of lattice sites. (This was first discussed in the non-random one-dimensional case by Haldane (1980).) These authors have suggested that the critical behaviour of the random system might be unconventional even in high dimensions.

The random spin problem was further investigated by Brackstone and Gunn (1987) who looked at the quantum fluctuations using a large-S expansion. They found that, for a fixed amount of disorder (measured by comparing the average magnitude of the random field with the exchange fields JS), the system has ferromagnetic order for a sufficiently large S. The large number of spin states $(2S+1)$ may correspond loosely to a weak repulsion in the Bose condensation problem so that many bosons may occupy each site. This is by no means a proper mapping because the spin-model analogue unnecessarily restricts the boson occupation at each site to a maximum of $2S+1$. It is therefore interesting to pursue the question of whether a suitable *weak-repulsion* limit with ODLRO exists in the boson problem.

We believe that the possibility of such a condensed limit in a disordered medium is associated with the *screening* behaviour of the system. In a naïve picture of screening, the first ^{4}He atoms to be added will plug up the bumpy landscape of the substrate, allowing additional atoms to travel over a much smoother environment. This argument often leads to the suggestion of an inert layer of localized atoms over which further atoms can undergo Bose condensation. We think that the Bose–Einstein statistics will not favour this assignment of distinct (localized and delocalized) rôles to the atoms. Fisher and Fisher (1988) have also suggested that the scaling behaviour of such a picture is inconsistent with experimental observations. Instead, we believe that the condensate itself will bulge and contract to perform the screening. In this paper, we will examine the extent to which this can be done.

We will concentrate on the case of two dimensions, although the concept of condensate screening is applicable in higher dimensions as well. The one-dimensional problem has been discussed by Giamarchi and Schulz (1987) using the Haldane representation (Haldane 1981) for the bosons. This is a special case because even the pure system at $T = 0$ has no finite condensate density. In fact, 'superfluidity' in one dimension is defined through power-law spatial correlations, analogous to the classical Kosterlitz–Thouless phase. For a system with short-range interaction, it is believed that the lowest dimension for the existence of genuine ODLRO at absolute zero is two for the pure case, based on the fact that the mean-field theories are at least stable to small zero-point fluctuations. This is because the effect of these fluctuations at long wavelengths becomes less important in higher dimensions. (Azbel (1980) argues that, for a charged system, there may be condensation in one dimension.) The marginal dimension for the existence of delocalized states in single-particle localisation theory also happens to be two. Therefore, the two-dimensional case seems the most interesting.

Using a loose analogy with spin models, we have already provided a theoretical motivation for finding a suitable 'non-interacting' limit where an extended Bose condensate exists (i.e. stable to zero-point fluctuations) even when the chemical potential is below the mobility edge of the *single-particle* spectrum. In the next section, we will also give a physical picture to suggest that this limit is achieved by the interaction vanishing and the density diverging such that the average of the repulsion at each site (i.e. the average Hartree potential) is kept fixed. We will then formalize the idea of screening described above. In section 3, we give some quantitative indication of the effectiveness of the screening in a lattice model with weak disorder. In section 4, we will discuss the results of the calculation and speculate on the case of strong disorder.

2. The condensed phase: general considerations

Although a gas of independent bosons is a bad starting point for the random problem, we will see that there is a *form* of non-interacting limit that is useful. To motivate this, let us consider the Anderson model for interacting bosons with short-range repulsion on a lattice:

$$H = -t \sum_{(\boldsymbol{n}\boldsymbol{n}')} c^\dagger_{\boldsymbol{n}} c_{\boldsymbol{n}'} + \sum_{\boldsymbol{n}} (\sigma V_{\boldsymbol{n}} - \mu) n_{\boldsymbol{n}} + \frac{1}{2} U \sum_{\boldsymbol{n}} n^2_{\boldsymbol{n}}. \tag{1}$$

The operators $c^\dagger_{\boldsymbol{n}}$ and $c_{\boldsymbol{n}}$ create and annihilate bosons at site $\boldsymbol{n}$. $V_{\boldsymbol{n}}$ is random with unit variance on each site and σ represents the magnitude of the random potential.

For simplicity, we will assume that the randomness is spatially uncorrelated, i.e. white noise.

Our point may be made by examining the case where the hopping, t, is set equal to zero. In this case each site 'fills up' with an integral number of bosons until the energy to add one more is above the chemical potential, μ. Let us assume that the disorder is weak so that the number of bosons on every site is much greater than unity. Then the resulting 'Hartree potential', $V_{\boldsymbol{n}} + Un_{\boldsymbol{n}}$, varies around the chemical potential from site to site by a fraction of the repulsive energy U (figure 1). This remaining variation (due to integral occupation numbers) would be smoother than the original disorder if the repulsion is weak compared to σ.

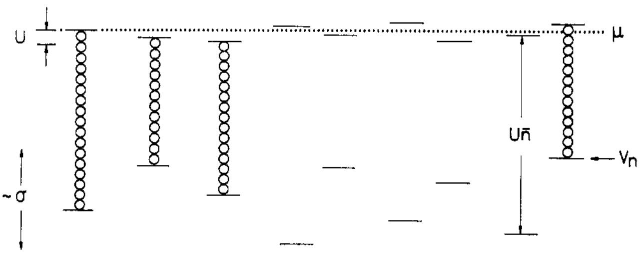

Figure 1. Screening at zero hopping and weak disorder.

To obtain a completely smooth potential, we can take the limit where $U \to 0$. Since we do not want to alter the source of the screening, namely the Hartree potential, we will also have to let the average site occupation $\overline{n}$ diverge such that the product $U\overline{n}$ is fixed. (Commensuration effects disappear in this limit.) Imagine that we now add an extra particle to the system, it will see a totally flat landscape, as in a homogeneous system. However, if we want to have a phase with an *extended* Bose condensate, we need to restore the hopping term. In fact, we will see that the screening becomes imperfect when there is finite hopping. Nevertheless, motivated by this screening argument, our strategy is to assume that a condensed phase exists and then to establish whether it is stable to zero-point fluctuations in our regime of a dense but weakly interacting gas.

Before we investigate the stability of the condensate, we need a quantitative picture of how the condensate adapts to the random potential. (For the homogeneous interacting Bose gas this stage would not exist—one would proceed immediately to the Bogoliubov transformation.) The interacting Bose gas has a characteristic healing length, $\lambda = (2ta^2/U\overline{n})^{1/2}$, over which the condensate wavefunction may vary significantly. (a is the lattice spacing.) We use this as our unit of length. In this section, we will use $U\overline{n}$ as our unit of energy. For the formal discussion in this section, it is more convenient to work in a continuum version of the problem. Assuming that each member of the grand canonical ensemble has a 'totally' condensed trial wave function Ψ for M particles of the form $\Psi = \prod_{i=1}^{M} \varphi(\boldsymbol{x}_i)$, the expectation value of the Hamiltonian is given by:

$$H[\varphi] = \overline{n}h[\varphi] = \overline{n} \int \left[\tfrac{1}{2}|\nabla\varphi|^2 + (\sigma V - \mu)|\varphi|^2 + \tfrac{1}{2}|\varphi|^4\right] \mathrm{d}^2\boldsymbol{x}. \qquad (2)$$

We have rescaled such that $\overline{n}^{1/2}\varphi$ is now the trial wavefunction. The condensate wavefunction φ_0 minimizes this Hamiltonian, so that it satisfies the nonlinear Schrödinger equation:

$$- \tfrac{1}{2}\nabla^2\varphi_0 + (\sigma V - \mu + \varphi_0^2)\varphi_0 = 0. \tag{3}$$

This ground-state wavefunction has no nodes. It should be extended because the energy penalty of the repulsive interaction forbids the bosons from condensing onto one region of the system. In the pure case, $\varphi_0 = 1$. In general, we can see from (3) that the condensate is unable to adjust to the variations in the random potential σV over a length scale smaller than the healing length λ. This feature is absent from the zero-hopping picture above. Consequently, instead of getting a totally smooth potential, we are left with some *residual* disorder even when $U \longrightarrow 0$. In other words, screening is ineffective when $t \gg U\overline{n}$.

It is interesting to observe that the chemical potential has to be decreased in order to keep the number of bosons constant. Dividing (3) by φ_0 and taking the spatial average, we see that the decrease in μ should be $-\int |\nabla\varphi_0|^2/\varphi_0^2 \, \mathrm{d}^2\boldsymbol{x}$.

We will now formalize our heuristic picture of the residual disorder in a simple Hartree picture. A natural choice for the residual potential is:

$$W = \sigma V - \mu + \varphi_0^2. \tag{4}$$

A single particle in this potential will have eigenstates φ_λ defined by the Schrödinger equation:

$$- \tfrac{1}{2}\nabla^2\varphi_\lambda + W\varphi_\lambda = \epsilon_\lambda\varphi_\lambda. \tag{5}$$

Note that the condensate wavefunction φ_0 is the zero-energy solution to this Schrödinger equation. The rationale behind this choice of W is based on the observation that the excitations of our interacting system should be orthogonal to the ground state. In our Hartree treatment here, a suitable basis set for these excitations should consist of wavefunctions orthogonal to the condensate. A natural basis set satisfying this orthogonality requirement is the set $\{\varphi_\lambda\}$, excluding φ_0. Therefore, our definition for W has the advantage of giving us a natural set of wavefunctions for the discussion of the excitations of the system.

As a matter of interest, we can ask what kind of excitation we are talking about when we add an extra particle to the system in our heuristic argument above. The excited state is in fact described by the unsymmetrized wavefunction $\phi(\boldsymbol{x}_{M+1})\prod_{i=1}^{M}\varphi_0(\boldsymbol{x}_i)$. The eigenstates $\phi = \varphi_\lambda$ (with energies ϵ_λ) are obtained by minimising the Hamiltonian with respect to ϕ and the Schrödinger equation obtained is the same as (5)! Therefore, our heuristic picture adds a distinguishable particle to the system. On the other hand, if our additional particle was identical with the rest, then we would have to use a symmetrized wavefunction. The resultant Schrödinger equation is similar to (5), with $W + \varphi_0^2$ replacing W. As already explained, the decision not to use this alternative form as the residual potential is a decision based on mathematical convenience only.

A qualitative indication of the smoothness of the residual disorder is to ask whether (5) gives us any states localized around a region of the lattice. First of all, we notice that the nonlinear Schrödinger equation yields an extended state at zero energy. We will show below that $\epsilon_\lambda \geq 0$ for all the basis functions φ_λ. Since these states lie above an extended state, we expect that they are not strongly localized, at least in the low-energy tail of the spectrum. To prove that the spectrum of eigenvalues are non-negative, we use a variational approach. Any time-independent trial wavefunction ϕ in (2) which is orthogonal to φ_0 can be written in the form:

$$\phi(\boldsymbol{x}) = \sum_\lambda{}' (c_\lambda + \mathrm{i} d_\lambda)\varphi_\lambda(\boldsymbol{x}). \tag{6}$$

The primed summation excludes $\lambda = 0$. The consequent quadratic deviation, δH, of the Hamiltonian $H[\phi]$ from its minimum value is given by the functional:

$$\begin{aligned}\delta H[\phi] &= \int \left[\phi^*(-\tfrac{1}{2}\nabla^2 + W)\phi + U\overline{n}\varphi_0^2\phi^*\phi + \tfrac{1}{2}U\overline{n}\varphi_0^2(\phi^2 + \phi^{*2})\right] \mathrm{d}^2\boldsymbol{x} \\ &= \boldsymbol{c}^{\mathrm{T}}(\boldsymbol{\epsilon} + 2U\mathbf{N})\boldsymbol{c} + \boldsymbol{d}^{\mathrm{T}}\boldsymbol{\epsilon}\boldsymbol{d} \end{aligned} \tag{7}$$

where we have written the real coefficients, c_λ and d_λ, as column vectors, $\epsilon_{\lambda\lambda'} = \epsilon_\lambda \delta_{\lambda\lambda'}$ and $N_{\lambda\lambda'} = \overline{n}\int \varphi_\lambda^* \varphi_0^2 \varphi_{\lambda'}\, \mathrm{d}^2\boldsymbol{x}$. The asymmetry between $\boldsymbol{c}$ and $\boldsymbol{d}$ arises because we have chosen φ_0 to be real. By the variational principle, δH cannot become negative for any choice of ϕ. By choosing $c = 0$ in particular, we can see that the second quadratic form in (7) has to be non-negative and we get the desired result that $\epsilon_\lambda \geq 0$ for all λ. Thus, we argue that the residual potential W does not have any localized states at the bottom of the spectrum. Since we are using these eigenstates as a basis for describing the excitations of our system, we can say that these excitations are also extended at low energies. In the final section, we will compare this problem with the case of amorphous materials where long-wavelength phonons see a nearly uniform effective medium.

The above result suggests that the smooth residual potential W does not have traps wide and deep enough to give rise to localized states. This is most apparent in the case of weak disorder ($\sigma \ll U\overline{n}$) where the variance of the Fourier component $W_{\boldsymbol{k}}$ has been reduced by a factor proportional to k^4 from that of the bare potential $V_{\boldsymbol{k}}$ for $k \ll 1$ (see next section). Therefore, the variation of W is small over long length scales so that large potential wells are likely to be shallow. There are deeper wells of smaller dimensions but the cost in kinetic energy for confining a particle there will be large. This picture at weak disorder makes the possible absence of low-energy localized states plausible.

Having found a natural basis set for the excitations, we will use it to look at the depletion of the condensate at zero temperature. The zero-point fluctuations involving these excitations involve exchange processes not included in the simplified discussion above. In particular, particles will be scattered into and out of the condensate. If the condensate indeed exists as we have assumed, then the fractional decrease in the number of particles in the condensate due to these processes will have to be small. We examine this condensate depletion in a second-quantized formulation of the problem. Write the field operators in terms of our basis:

$$\hat{\varphi}(\boldsymbol{x}) = \overline{n}^{1/2}\varphi_0(\boldsymbol{x}) + \sum_\lambda{}' \varphi_\lambda(\boldsymbol{x})\hat{c}_\lambda. \tag{8}$$

We have ignored the operator nature of $c_{\lambda=0}$ and replaced it by its expectation value $\overline{n}^{1/2}$. This is the Bogoliubov approximation which is justified in our condensed limit with a *large* number of bosons in the condensate. This is most apparent in the number-phase representation of the boson operators which we discuss in the next section.

The Hamiltonian can be organized into terms with an increasing number of operators with coefficients that have decreasing powers of $\overline{n}$. Following the Bogoliubov procedure, we will keep only terms up to second order in the operators. The Hamiltonian becomes

$$H = \overline{n}h[\varphi_0] + \sum_{\lambda\lambda'}{}' \left[(\epsilon_{\lambda\lambda'} + UN_{\lambda\lambda'})c^\dagger_\lambda c_{\lambda'} + \tfrac{1}{2}UN_{\lambda\lambda'}(c^\dagger_\lambda c^\dagger_{\lambda'} + c_\lambda c_{\lambda'})\right] + \cdots . \tag{9}$$

The rest of terms are $\sim \mathrm{O}(1/\overline{n})$. As $\overline{n} \to \infty$, it is clear that the Hamiltonian is dominated by the condensed term $\overline{n}h[\varphi_0]$ in which case the zero-point harmonic fluctuations should have vanishing importance. Thus, our limit is a 'classical' one, leading us to a stable condensate. (Compare with the $S \to \infty$ limit in spin systems.)

However, before making such a claim, we should consider the effect of the zero-point fluctuations more carefully. We can diagonalize the harmonic part of the Hamiltonian by a Bogoliubov transformation:

$$\gamma^\dagger_\mu = \sum_\lambda{}' (u^\lambda_\mu c^\dagger_\lambda + v^\lambda_\mu c_\lambda) \tag{10}$$

with $\mathbf{u}\mathbf{u}^{\mathrm{T}} - \mathbf{v}\mathbf{v}^{\mathrm{T}} = \mathbf{1}$ and $\mathbf{u}\mathbf{v}^{\mathrm{T}} - \mathbf{v}\mathbf{u}^{\mathrm{T}} = 0$ to preserve the commutation relations for bosons for the new operators: $[\gamma_\mu, \gamma^\dagger_{\mu'}] = \delta_{\mu\mu'}$. In a first-quantized formulation, the above procedure corresponds to finding the normal modes of the condensate. The solutions, labelled by μ, are:

$$\phi_\mu(\boldsymbol{x}, t) = \sum_\lambda{}' [u^\lambda_\mu \varphi_\lambda(\boldsymbol{x}) \exp(-\mathrm{i}E_\mu t) + v^\lambda_\mu \varphi^*_\lambda(\boldsymbol{x}) \exp(\mathrm{i}E_\mu t)]. \tag{11}$$

The normalisation of ϕ will then give us the conditions on $\mathbf{u}$ and $\mathbf{v}$.

The harmonic Hamiltonian, diagonalized in terms of the new operators, has energy eigenvalues $E = E_\mu$ satisfying the the secular equation:

$$\det \begin{vmatrix} \epsilon + U\mathbf{N} - E & U\mathbf{N} \\ U\mathbf{N} & \epsilon + U\mathbf{N} + E \end{vmatrix} = 0. \tag{12}$$

We can see that this Bogoliubov spectrum depends on the dimensionless parameters $t/U\overline{n}$ and $\sigma/U\overline{n}$. The change of sign of E in the determinant is related to the relationship between $\mathbf{u}$ and $\mathbf{v}$.

What are the low-energy excitations of the system? In the *pure* case, it is well known that these are long-wavelength phase fluctuations. This originates from the fact that a uniform phase change in $\varphi = \varphi_0 = 1$ will not alter the expectation value $h[\varphi]$. This continuous symmetry remains intact in our random system so that we should have a gapless spectrum, by the Goldstone theorem. As in the pure case, these collective phase fluctuations at low energies should be the dominant factor accounting for the zero-point depletion of the condensate. Following conventional nomenclature, we will henceforth call these excitations 'phonons'.

If the condensate is indeed stable with respect to the fluctuations, the single-particle density matrix $\langle c^\dagger_{\boldsymbol{n}} c_{\boldsymbol{n}'}\rangle$ should not have more than one macroscopically large eigenvalue (Yang 1962). In our φ_λ representation, the appropriate matrix is $\langle c^\dagger_\lambda c_{\lambda'}\rangle$.

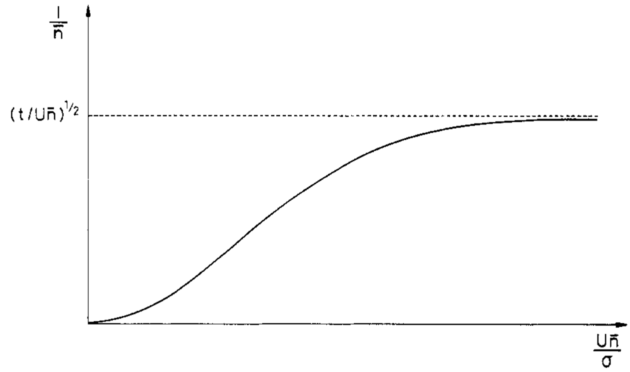

Figure 2. Schematic phase diagram in the parameter space defined by $1/\overline{n}$ and $U\overline{n}/\sigma$. $t/U\overline{n}$ is fixed. We have extrapolated our weak-disorder calculation for the entire phase boundary which is defined by the criterion $\Xi(\sigma) = \overline{n}$.

The depletion Ξ is measured by the number of particles not in the condensate (Gross 1966):

$$\Xi = \sum_{\lambda}{}' \langle c_{\lambda}^{\dagger} c_{\lambda} \rangle = \sum_{\mu\lambda}{}' |v_{\mu}^{\lambda}|^2. \tag{13}$$

Note that Ξ is a function of the dimensionless ratios $t/U\overline{n}$ and $\sigma/U\overline{n}$. Extrapolating from the pure case, we expect that Ξ increases when $t/U\overline{n}$ is decreased. It also seems sensible that the randomness will give rise to more particles being scattered out of the condensate so that the depletion increases when $\sigma/U\overline{n}$ is increased. When the fractional depletion $\Xi/\overline{n}$ reaches unity, the condensate will no longer exist and the system will have had a transition to an uncondensed phase such as the localized Bose glass.

In the pure case, the transition occurs when $(U\overline{n}/t)^{1/2} \sim \overline{n}$. Therefore, the stability of the homogeneous condensate can always be ensured in our 'classical' $\overline{n} \to \infty$ limit. It is clear that the depletion Ξ would have to *diverge* to indicate an instability to a phase transition in this limit. In the pure case, a likely physical origin of such a singularity is a vanishing group velocity for the long-wavelength quasiparticles—'phonons'. Can this case arise in the random system? As $\overline{n} \to \infty$, the Hamiltonian is dominated by the condensate term $\overline{n}h[\varphi_0]$ and this term gives a compressional 'sound' velocity of the condensate which would be a good approximation for the group velocity in question. Since the compressibility of the condensate itself should be finite even in the disordered case, we expect that the low-lying excitations should have a finite (if small) group velocity in the classical limit. Another possible source of a divergence in Ξ is the density of states of these excitations. However, since we have suggested that the low-lying states are not strongly localized, it is unlikely that there are any radical differences between the pure and disordered cases which leads to a non-integrable singularity in the summation in Ξ.

We have now arrived at the conclusion that $\Xi/\overline{n} \to 0$ in the dense limit for any *finite* $t/U\overline{n}$ and $\sigma/U\overline{n}$ because Ξ can only have a singularity at infinity as a function of $U\overline{n}/t$ or $\sigma/U\overline{n}$. In other words, there is always a finite interval $0 < 1/\overline{n} < 1/\overline{n}_c(t/U\overline{n}, \sigma/U\overline{n})$ where Bose condensation is stable as long as neither the disorder nor the healing length is infinite (see figure 2). Therefore, condensation may exist even when the chemical potential is far below any single-particle mobility edge of the original random potential.

Provided that the critical velocity is non-zero (not the case for a perfect Bose gas), the existence of such an extended condensate φ_0 indicates superfluidity in our system, as defined by the quantisation of circulation (see e.g. Leggett 1973).

3. Screening in the lattice model

In the condensed phase, we have argued that the system has a vanishing depletion of the condensate by virtue of good screening in the limit of $U \rightarrow 0$ and $\overline{n} \rightarrow \infty$ with $U\overline{n}$ fixed. We have also pointed out that the screening should be good when $t \ll U\overline{n}$ (small bandwidth) and ineffective when $t \gg U\overline{n}$ (large bandwidth). Let us now examine this in more detail. We will concentrate on the lattice Hamiltonian of (1) in the case of weak disorder, i.e. $\sigma \ll U\overline{n}$. Since the fractional depletion is small in this dense limit, the Bogoliubov approximation is valid. The effect of disorder will be incorporated perturbatively as the scattering processes of the Bogoliubov quasiparticles and the condensate particles. We will ignore other processes such as disintegration and recombination. This parallels the Hartree treatment in the previous section. From now on, $2U\overline{n}$ is used as the unit of energy.

To illustrate our previous arguments, we give an estimate of the condensate depletion from the limit of $\langle c^{\dagger}_{\boldsymbol{n}} c_{\boldsymbol{n}'} \rangle$ at large separation. We have argued in the last section that the low-lying excitations are not strongly localized, we can try to calculate the group velocity c in the long-wavelength limit. These 'phonons' should consist of phase fluctuations (and weak density fluctuations to satisfy the number-phase uncertainty relation) giving rise to the depletion of the condensate. In the *pure* case, c is given by $t^{1/2}$ and the fractional depletion of the condensate is $(2\omega_{\mathrm{B}}^{1/2}\overline{n})^{-1}$ for $\omega_{\mathrm{B}} \ll 1$ and $(4\omega_{\mathrm{B}}\overline{n})^{-1}$ for $\omega_{\mathrm{B}} \gg 1$ where $\omega_{\mathrm{B}} = 2zt$ is the bandwidth of the lattice model. (z is the number of nearest neighbours.) We see that the decrease in c is accompanied by an increase in the depletion. Indeed, the 'softening' of these phonon modes eventually gives us a transition to an uncondensed phase. We would like to see how this carries over into the disordered system and so it should be interesting to investigate this issue in our perturbative model.

In addition to phonons, the spectrum may also have a part at higher energies which is mainly single particle in character (in the Bogoliubov approximation). Since it is natural to deal initially with a spectrum with only one type of behaviour, we will focus our discussion on the case of $\omega_{\mathrm{B}} \ll 1$ where the spectrum is mainly collective. Moreover, the effect of disorder on the single-particle excitations should be small because their kinetic energies are high compared with the variations in the random potential ($\sim \sigma$).

In order to make the importance of the collective phase fluctuations manifest, we will work in the number-phase representation: $c^{\dagger}_{\boldsymbol{n}} \leftrightarrow n_{\boldsymbol{n}}^{1/2} \exp(\mathrm{i}\phi_{\boldsymbol{n}})$. This representation is useful in our case because the existence of a condensate brings phase coherence so that the fluctuations in the phase of the order parameter is small compared to 2π. Moreover, the fractional fluctuations in the boson number would be small in our dense $\overline{n} \rightarrow \infty$ limit. In the pure case, for example, the fractional number fluctuation on each site is $\sim \mathrm{O}(1/\overline{n}^{1/2})$. Hence, the operators can be defined by suitable Taylor expansions. In particular, the square root of the number operator is defined by a Taylor expansion around its c-number part i.e. its condensate value, $\widetilde{n}_{\boldsymbol{n}}$.

Substituting just this c-number leading term in that expansion into (1), we may minimize the Hamiltonian with respect to $\widetilde{n}_{\boldsymbol{n}}$. We obtain a lattice version of the nonlinear Schrödinger equation:

$$-t\sum_{n'}(\tilde{n}_{n'}/\tilde{n}_{n})^{1/2}+\sigma V_{n}-\mu+U\tilde{n}_{n}=0 \tag{14}$$

where we have summed over the nearest neighbours n' of site n. Hence, $\tilde{n}^{1/2}$ corresponds to the condensate wavefunction.

In the pure case, $\tilde{n}_{n}=\overline{n}$ where $U\overline{n}=\mu+zt$. For weak disorder, we can write $\tilde{n}_{n}=\overline{n}(1+\sigma\nu_{n})+O(\sigma^2)$ and obtain the fractional deviation, $\sigma\nu_{n}$, from the pure case as a perturbation in powers of σ. Defining Fourier transforms such as ν_{k} to be $N^{-1/2}\sum_{n}\nu_{n}\exp(-\mathrm{i}k\cdot n)$ into (14), we get the first-order correction as:

$$\nu_{k}=-2V_{k}/(\omega_{k}+1) \tag{15}$$

where $\omega_{k}=2t\sum_{\delta}\sin^2(k\cdot\delta/2)$ is the tight-binding spectrum. (δ is any vector joining nearest neighbours.) Note that ν_{k} is small for $\omega_{k}\gg 1$ that the condensate wavefunction does not adjust to the potential on a length scale shorter than the healing length. The variance of the residual potential as defined in (4) has the behaviour $\overline{|W_{k}|^2}\sim\sigma^2\omega_{k}^2$, as mentioned in the last section.

At the second order in the randomness, we have to lower the chemical potential to keep the overall number density unchanged:

$$\delta\mu=-\frac{\sigma^2}{2N}\sum_{k}\omega_{k}|\nu_{k}|^2\simeq\begin{cases}-\sigma^2\omega_{\mathrm{B}} & \text{for } \omega_{\mathrm{B}}\ll 1\\ -2\sigma^2\ln\omega_{\mathrm{B}}/\omega_{\mathrm{B}} & \text{for } \omega_{\mathrm{B}}\gg 1.\end{cases} \tag{16}$$

This correction has already been pointed out in the last section.

Let us now examine the zero-point fluctuations in the system. From now on, we will discuss only the low-lying collective excitations in the case of $\omega_{\mathrm{B}}\ll 1$, for the reasons already mentioned. It is convenient to use the rescaled (real) operators Δ and Φ as defined by

$$n_{n}=\tilde{n}_{n}+(2\overline{n})^{1/2}\Delta_{n}\qquad\phi_{n}=\Phi_{n}/(2\overline{n})^{1/2}. \tag{17}$$

Note that Δ and Φ are conjugate variables: $[\Delta_{n},\Phi_{n'}]=-\mathrm{i}\,\delta_{nn'}$. From this, we can now define 'number-phase bosons' by the creation and annihilation operators $a_{n}^{\dagger}$ and a_{n}:

$$a_{n}^{\dagger}=(\Delta_{n}+\mathrm{i}\,\Phi_{n})/\sqrt{2}\qquad a_{n}=(\Delta_{n}-\mathrm{i}\,\Phi_{n})/\sqrt{2} \tag{18}$$

satisfying $[a_{n},a_{n'}^{\dagger}]=\delta_{nn'}$.

In the homogeneous case, it is easy to check that the spectrum calculated using this representation is the same as that calculated *via* the conventional Bogoliubov method which uses the bosons operators c_{n} directly. Moreover, to order $1/\overline{n}$, the effect of $a^{\dagger}$ is the same as $c^{\dagger}$. (The casual treatment of the annihilation and creation operators of the condensed ($k=0$) state as c-numbers is a surreptitious use of the leading order approximation of this representation.) One appealing feature of the number-phase representation is that we can borrow the language of Josephson junctions for our bulk case. More profoundly, the dense limit allows one to discuss condensation in k-space conveniently in a basis set which is based in real space. This is because the fractional fluctuation on each site is small so that, unlike the dilute gas, there are always a large number of particles on each site, giving us a well defined $\tilde{n}_{\boldsymbol{n}}$. Thus we may handle disorder, naturally expressed in real space, along with condensation in k-space. Moreover, it is the long-range *phase*-order which is associated with condensation and this is naturally discussed in the number-phase representation.

To lowest order in $1/\overline{n}$, we need only treat the terms quadratic in quantum fluctuations, which are scattering processes. Truncated at this level, the pure Hamiltonian, H_0, is diagonalized by the Bogoliubov transformation:

$$a_{\boldsymbol{k}} = u_{\boldsymbol{k}}\alpha_{\boldsymbol{k}} - v_{\boldsymbol{k}}\alpha^{\dagger}_{-\boldsymbol{k}} \qquad a^{\dagger}_{\boldsymbol{k}} = u_{\boldsymbol{k}}\alpha^{\dagger}_{\boldsymbol{k}} - v_{\boldsymbol{k}}\alpha_{-\boldsymbol{k}}. \tag{19}$$

The coherence factors can be written as $u_{\boldsymbol{k}} = \cosh\theta_{\boldsymbol{k}}$ and $v_{\boldsymbol{k}} = \sinh\theta_{\boldsymbol{k}}$ where $\theta_{\boldsymbol{k}} > 0$. H_0 has become

$$\sum_{\boldsymbol{k}} \Omega_{\boldsymbol{k}}(\alpha^{\dagger}_{\boldsymbol{k}}\alpha_{\boldsymbol{k}} + \tfrac{1}{2}) \qquad \Omega^2_{\boldsymbol{k}} = \omega_{\boldsymbol{k}}(\omega_{\boldsymbol{k}} + 1) \tag{20}$$

where $\exp(2\theta_{\boldsymbol{k}}) = \Omega_{\boldsymbol{k}}/\omega_{\boldsymbol{k}}$. The slope of $\Omega_{\boldsymbol{k}}$ in the long-wavelength limit is the phonon velocity, c.

In the number-phase representation, the residual disorder is in fact off-diagonal, appearing in the hopping part of the Hamiltonian. (Diagonal disorder will be present if we also have a random U but it does not alter the scattering matrix elements below in a qualitatively significant manner.)

It is expressed in terms of the spatial variation $\sigma\nu_{\boldsymbol{k}}$ of the condensate. The consequent scattering processes, at the leading order in the randomness, are given by

$$\begin{aligned} H_1 &= \frac{\sigma}{N}\sum_{\boldsymbol{k}\neq\boldsymbol{q}}\left(S_{\boldsymbol{kq}}\alpha^{\dagger}_{\boldsymbol{k}}\alpha_{\boldsymbol{q}} + S^{*}_{\boldsymbol{kq}}\alpha_{\boldsymbol{k}}\alpha^{\dagger}_{\boldsymbol{q}} + T_{\boldsymbol{kq}}\alpha^{\dagger}_{\boldsymbol{k}}\alpha^{\dagger}_{\boldsymbol{q}} + T^{*}_{\boldsymbol{kq}}\alpha_{\boldsymbol{k}}\alpha_{\boldsymbol{q}}\right) \\ S_{\boldsymbol{kq}} &= \left[\sinh(\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{q}})(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{q}}) - \cosh(\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{q}})\omega_{\boldsymbol{k}-\boldsymbol{q}}\right]\nu_{\boldsymbol{k}-\boldsymbol{q}} \\ T_{\boldsymbol{kq}} &= \left[\sinh(\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{q}})\omega_{\boldsymbol{k}-\boldsymbol{q}} - \cosh(\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{q}})(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{q}})\right]\nu_{\boldsymbol{k}+\boldsymbol{q}}. \end{aligned} \tag{21}$$

We also have an additional *non-random* part, H_2. This comes from the coefficient of the $\Delta^2_{\boldsymbol{n}}$ terms from the number-phase expansion of the hopping part of the Hamiltonian. Since we are interested in the low-lying collective excitations, we will just look at the low $\boldsymbol{k}$ contribution which is most conveniently written as:

$$H_2 \simeq \sigma^2 \sum_{\boldsymbol{k}} 2|\delta\mu|\Delta_{\boldsymbol{k}}\Delta_{-\boldsymbol{k}}. \tag{22}$$

The reason for keeping these second-order terms will shortly become clear.

We perform a unitary transformation e^{-A} on the Hilbert space to eliminate the terms linear in the disorder $\sigma\nu$. Under this transformation, the Hamiltonian, H, becomes $H' = \mathrm{e}^{-A}H\mathrm{e}^{A}$. The appropriate choice satisfies $[H_0, A] = -H_1$. Hence,

$$\begin{aligned} A = -\frac{\sigma}{\sqrt{N}}\sum_{\boldsymbol{k}\neq\boldsymbol{q}}\Bigg(&\frac{S_{\boldsymbol{kq}}}{\Omega_{\boldsymbol{k}} - \Omega_{\boldsymbol{q}}}\alpha^{\dagger}_{\boldsymbol{k}}\alpha_{\boldsymbol{q}} - \frac{S^{*}_{\boldsymbol{kq}}}{\Omega_{\boldsymbol{k}} - \Omega_{\boldsymbol{q}}}\alpha_{\boldsymbol{k}}\alpha^{\dagger}_{\boldsymbol{q}} \\ &+\frac{T_{\boldsymbol{kq}}}{\Omega_{\boldsymbol{k}} + \Omega_{\boldsymbol{q}}}\alpha^{\dagger}_{\boldsymbol{k}}\alpha^{\dagger}_{\boldsymbol{q}} - \frac{T^{*}_{\boldsymbol{kq}}}{\Omega_{\boldsymbol{k}} + \Omega_{\boldsymbol{q}}}\alpha_{\boldsymbol{k}}\alpha_{\boldsymbol{q}}\Bigg). \end{aligned} \tag{23}$$

The treatment of the $\Omega_k = \Omega_q$ terms will be discussed presently. Then, the Hamiltonian becomes

$$H' = H_0 + H_2 + [H_1, A]/2 + \cdots \tag{24}$$

while each state $|\Psi\rangle$ in the Hilbert space becomes $|\Psi'\rangle = \mathrm{e}^{-A}|\Psi\rangle$.

Now that the Hamiltonian has become second order in the randomness, we will average over the disorder. (We can now see why the second-order non-random term H_2 has been retained.) Diagrammatically, the averaging will include only the minimally crossed terms in the self energy of the quasiparticles. This approximation breaks down as we approach the transition where condensation is destroyed. However, since we believe that a condensed phase does exist in two dimensions, the averaged Hamiltonian, $\overline{H}$, should still have physical interest in such a phase. The first two terms in H are not random and the last term, $[H_1, A]/2$, averages to

$$\overline{H}_3 = \frac{\sigma^2}{N}\sum_{kq}\left[2\left(\frac{\overline{|S_{kq}|^2}}{\Omega_k - \Omega_q} - \frac{\overline{|T_{kq}|^2}}{\Omega_k + \Omega_q}\right)(\alpha_k^\dagger\alpha_k + \alpha_k\alpha_k^\dagger)\right.$$
$$\left. + 4\overline{S_{kq}T_{q-k}}\frac{\Omega_q}{\Omega_k^2 - \Omega_q^2}(\alpha_k^\dagger\alpha_{-k}^\dagger + \alpha_k\alpha_{-k})\right]. \tag{25}$$

The average Hamiltonian $\overline{H} = H_0 + H_2 + \overline{H}_3$ can be diagonalized by a second Bogoliubov transformation:

$$\alpha_k = u'_k\beta_k - v'_k\beta_{-k}^\dagger \qquad \alpha_k^\dagger = u'_k\beta_k^\dagger - v'_k\beta_{-k}$$

where we can write $u'_k = \sinh\theta'_k$ and $v'_k = \cosh\theta'_k$. v'_k should be of the order of σ^2 and u'_k approximately unity. The ground state of the averaged Hamiltonian in the new Hilbert space is defined by

$$\beta_k|0'\rangle = 0.$$

We now return to consider the definition of the transformation and the transformed Hamiltonian where there are terms with vanishing denominators. The scattering matrix elements have simple poles at $\Omega_k = \Omega_q$ because some virtual processes involve the normal scattering of state k into a state q with the same energy. These degeneracies give rise to a finite lifetime, $\tau_k = 2\Gamma_k^{-1}$, for the quasiparticles. The decay rate Γ_k is the imaginary part of $\widetilde{\Omega}_k$ obtained by separating out the principal part of the sum in $\overline{H}$ across the poles. This is identical with a conventional calculation *via* Fermi's Golden rule. We estimate that

$$\Gamma_k/\Omega_k \simeq \sigma^2k^2/2 \qquad \text{for } \omega_k \ll 1. \tag{26}$$

Since these decay rates are small for weak disorder, we should be safe in ignoring them. It should be noted that decay due to any nonlinear processes is not included.

Let us examine the modification to the low-lying states, labelled by small k vectors. We will use for simplicity a constant density of states for the non-interacting spectrum (ignoring the logarithmic singularity which occurs at midband for a square lattice).

The perturbed spectrum, $\tilde{\Omega}_{\boldsymbol{k}}$, gives us an altered phonon velocity. Compared with the pure case, c has changed by a factor of $(1-\sigma^2)^{1/2}$ for $\omega_{\mathrm{B}} \ll 1$.

What is the depletion of the system described by $\overline{H}$? We estimate this from the limit of $\langle c_{\boldsymbol{n}}^{\dagger} c_{\boldsymbol{n}'}\rangle$ at large separation. The number of bosons not in the condensate should then be given by

$$\begin{aligned}\sum_{\boldsymbol{k}}{}' \langle c_{\boldsymbol{k}}^{\dagger} c_{\boldsymbol{k}}\rangle &= \frac{1}{N}\sum_{\boldsymbol{knn}'}{}' \langle n_{\boldsymbol{n}}^{1/2} \mathrm{e}^{\mathrm{i}(\phi_{\boldsymbol{n}}-\phi_{\boldsymbol{n}'})} n_{\boldsymbol{n}'}^{1/2}\rangle \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{n}-\boldsymbol{n}')} \simeq \sum_{\boldsymbol{k}}{}' \langle a_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}}\rangle \\ &= \sum_{\boldsymbol{k}}{}' \langle u_{\boldsymbol{k}}^2 \alpha_{\boldsymbol{k}}^{\dagger}\alpha_{\boldsymbol{k}} + v_{\boldsymbol{k}}^2 \alpha_{-\boldsymbol{k}}\alpha_{\boldsymbol{k}}^{\dagger} - u_{\boldsymbol{k}} v_{\boldsymbol{k}}(\alpha_{-\boldsymbol{k}}\alpha_{\boldsymbol{k}} + \alpha_{\boldsymbol{k}}^{\dagger}\alpha_{-\boldsymbol{k}}^{\dagger})\rangle . \end{aligned} \tag{27}$$

(The primed summations exclude the $\boldsymbol{k} = 0$ contribution.) This can be expressed further in terms of the operators $\beta_{\boldsymbol{k}}^{\dagger}$ and $\beta_{\boldsymbol{k}}$ defined in the second Bogoliubov transformation. The expectation values in the ground state can be calculated using (25). For example, we need

$$\langle 0|\beta_{\boldsymbol{k}}\beta_{\boldsymbol{k}}^{\dagger}|0\rangle = \langle 0'|\mathrm{e}^{-A}\beta_{\boldsymbol{k}}\beta_{\boldsymbol{k}}^{\dagger}\mathrm{e}^{A}|0'\rangle .$$

Collecting together the second-order terms in σ, we obtain for the depletion:

$$\sum_{\boldsymbol{k}}{}' \langle a_{\boldsymbol{k}}^{\dagger} a_{\boldsymbol{k}}\rangle = \sum_{\boldsymbol{k}}{}' \sinh^2(\theta_{\boldsymbol{k}} + \theta_{\boldsymbol{k}}') + \frac{4\sigma^2}{N}\sum_{\boldsymbol{kq}}{}' \frac{2|T_{\boldsymbol{kq}}|^2}{(\Omega_{\boldsymbol{k}} + \Omega_{\boldsymbol{q}})^2} + \mathrm{O}(\sigma^4). \tag{28}$$

The first term shows the combined effect of the two Bogoliubov transformations. The second term comes from the application of (25) and is the effect of the unitary transformation that we employed to eliminate disorder to lowest order. Since the θ parameters of both Bogoliubov transformations are positive, we can see that the depletion is increased—by a factor of

$$1 + C(\sigma/2U\overline{n})^2 \qquad \text{for } \omega_{\mathrm{B}} \ll 1 \tag{29}$$

where $C = \frac{3}{2} - 2\ln 2$ is a constant of the order of unity.

We have shown in this section that the RPA treatment of the dense condensed limit is well behaved for the case of small bandwidth and good screening. While the depletion increases with a decreasing bandwidth, the condensate can adjust better to the bare potential. Thus, the screening capability improves and the change in the contribution of the low-lying collective excitations to the depletion due to disorder can be small. It can be seen that the depletion does not scale with the phonon velocity as in the pure case. Nevertheless, an increase in depletion is still associated with a decrease in the phonon velocity.

4. Discussion

Interaction between the bosons has two conflicting rôles in determining their response to a random potential. If there are no repulsive interactions, then the ground state and low-lying excited states are localized (Hertz *et al* 1980). However if the interactions are too strong then the bosons localize to form a Mott insulator (if there is an integral number per site). In both cases the extended condensate is destroyed. An intermediate regime is required for the stability of the condensate by allowing it to screen the random potential. Our $U \to 0$ and $\overline{n} \to \infty$ limit should serve this purpose.

Associated with the non-zero compressibility, necessary for screening, is the transformation of the excited states into collective modes—at least at low energies. It is interesting to ask what relationship our problem has with other localisation problems where the low-energy modes have a similar collective nature with linear dispersion relations, for instance light and sound (for a review see John 1988). The randomness is incorporated as white noise in the properties determining propagation such as the dielectric function for light and elastic constants for sound. Field-theoretic arguments have been used to indicate that, in two dimensions, even low-frequency waves are weakly localized with a localisation length diverging as e^{1/ω^2} as $\omega \to 0$ (see John 1985 and references therein). However, we cannot deduce this for our problem as the wave equation for the collective variable, $\phi(\boldsymbol{x})$, has a different form from the ordinary wave equation. This equation is most conveniently expressed in terms of the rescaled variables $\Phi' = \sqrt{2}\varphi_0^{1/2}\phi$:

$$\left[\frac{\partial^2}{\partial t^2} + \left(-\tfrac{1}{2}\nabla^2 + W + 2\varphi_0^2\right)\left(-\tfrac{1}{2}\nabla^2 + W\right)\right]\Phi' = 0. \tag{30}$$

This difference makes our comparison incomplete as we do not know if this equation is in the same universality class as the wave localisation equation considered by John (1985). Futhermore depletion of the condensate has no clear parallel in the light and sound cases. The closest phenomenon would be a divergent Debye–Waller factor in the latter case; we know of no calculations in this direction.

If we move away from the collective limit, what is the effect of letting the more highly excited states become single-particle in nature? Qualitatively it is expected that the screening will become worse as t is increased compared to U. One difficulty that such a calculation will encounter, on a technical level, is 'ultraviolet' problems in the simple perturbative scheme (Thouless 1974)—at least if a white noise potential is used. This occurs near the upper edge of the Bogoliubov spectrum which has retained much of the single-particle properties of the non-interacting case. This problem can be avoided if a finite spatial correlation length is introduced into the random potential, i.e. coloured noise. We have not done so here because we are interested in looking at a system dominated by the *collective* excitations.

In this paper so far, we have concentrated on comparisons between the bosonic and single-particle localisation problems. Another comparison has been made in the literature: that of percolation with the condensation transition at finite temperature (e.g. Fishman and Ziman 1982). The relation comes from a physical picture of regions locally 'condensing' as the temperature is lowered, the regions growing until they percolate.

We will now argue that even at zero temperature there is a limit in which the problem of the existence of the condensate is of a percolative nature. Amongst other things, we require that we have strong disorder in the sense that the chemical potential is deep in the Lifshitz tail of the bare potential σV, i.e. $\mu < 0, |\mu|/\sigma > 1$ and that $\sigma \gg t$. Some other conditions will be specified presently. At strong disorder, there are particles in only a fraction of the lattice sites. The wavefunction φ_0 has exponential tails outside these dense 'lakes' but remains extended. In our dense limit, there are a large number of bosons even in the tails. This offers the hope of retaining phase coherence among the lakes. Consider the case when $\sigma \sim \mathrm{O}(U\overline{n})$. This gives us a lattice coverage where two lakes are typically separated by a site where σV_n is very large. In simple perturbation theory, the effective hopping t_{eff} between the two lakes is approximately $(t\overline{n}^{1/2})^2/\sigma \propto t^2\overline{n}$ since $\sigma \sim U\overline{n}$ is fixed. For a given t, we will not have to worry about this as a weak coupling regime in our dense limit so that the phase coherence is always ensured across the system, as discussed in section 2.

To obtain the percolation limit, we construct a more tenuous scenario where the effective hopping between the lakes remains small. This is achieved by decreasing the hopping such that $t \to t/\overline{n}^{\epsilon+1/2}$ while $\overline{n} \to \infty$. Although this increases the overall depletion Ξ, the fractional depletion $\Xi/\overline{n}$ will scale as $1/t^{1/2}\overline{n} \sim \overline{n}^{\epsilon-3/4}$. Therefore, the condensate can be maintained as a network of weakly coupled lakes provided that $0 < \epsilon < \frac{3}{4}$. If we further require the degeneracy temperature $T_d \sim t\overline{n}$ (in two dimensions) to be finite, then we should also have $\epsilon \leq \frac{1}{2}$. These Josephson-like links have random strengths so that the phase coherence across the entire system should be determined by a *percolative* criterion. Thus we would expect the condensate to be destroyed when the lattice coverage, controlled by the ratio σ/μ, falls below the percolation threshold. This occurs when $\sigma/\mu \simeq \sigma/U\overline{n} \sim O(1)$. In other words, we have proposed a way in which t may vanish (together with $U \to 0, \overline{n} \to \infty$) such that the destruction of condensation is determined by *geometrical* factors rather than the dynamical considerations presented in the previous sections.

In summary, the $U \to 0$ and $\overline{n} \to \infty$ limit of our problem gives us an extended ground state φ_0. We have exploited this to find a regime of robust condensation. In this regime (small U and large $\overline{n}$ with $U\overline{n}$ fixed), the depletion of the Bose condensate is small. We attribute this to the effective screening of the random potential by the condensate. In particular, good screening is achieved when the healing length λ is short, i.e. $t \ll U\overline{n}$. Using this argument, a RPA calculation of the effect of weak disorder has been set up to illustrate our point. This screening behaviour is a new factor in the *interacting* random problem that is absent from the single-particle theory. It is clearly responsible for the possibility of a simple perturbative scheme for the disorder in two dimensions, at least in the regime we have considered. Physically, this has brought about Bose–Einstein condensation even when μ is in the Lifshitz tail of the bare potential.

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