Quantum Theory of Matter

Revision Lecture

Derek Lee

Imperial College London

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Outline

Outline

1. Exam and Revision
2. Quantum Theory of Matter
   - Phenomenology
   - Microscopic theory
3. Summary

Exam

- Exam format
  - Two hours
  - 2 sections with 3 questions each
    - section A: phenomenology (chap 2 to 7); concentrates on understanding of basic phenomena.
    - section B: microscopic theory (chap 1, 8 to 12); involves a few questions with more technical content.
  - Answer 3 questions: **at least one question from each section**

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  - Answer 3 questions: **at least one question from each section**
- Exam technique
  - take 5 minutes to read all the questions before you start
  - check your algebra by checking the dimensions of your answers at each step

Revision

- Lecture notes
  - your own notes
  - my typed notes: extended description of lecture contents
- Lecture summaries: use as reminder of key points
- Problem sheets: do the problems!
  - in particular, key concepts are covered in questions assigned for **rapid feedback** (listed on course web page)
  - do not to refer to the solutions unless you are completely stuck
- Office hours: email to arrange meeting

Outline

Superfluidity

- Superfluids flow without viscosity
  - neutral: liquid helium ^{4}He (below 2K), optically trapped alkali atoms (nK)
  - charged: superconductors $\approx$ Cooper pairs (charge $2e$, mass $m^* = 2m_e$)

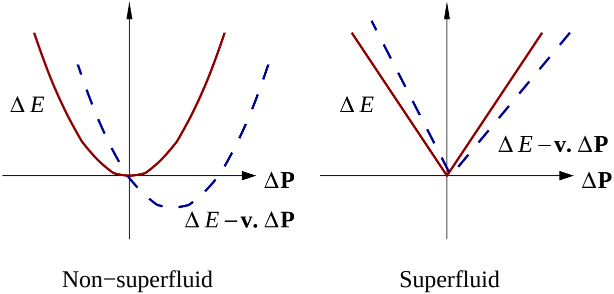
Superfluidity

- Superfluids flow without viscosity
  - neutral: liquid helium ^{4}He (below 2K), optically trapped alkali atoms (nK)
  - charged: superconductors ≈ Cooper pairs (charge $2e$, mass $m^* = 2m_e$)
- Landau criterion: can the fluid lose momentum/energy due to scattering with the container wall when travelling at a small relative velocity $\mathbf{v}$?
  - inspect excitations in the rest frame: energy $\Delta E(\mathbf{p})$ and momentum $\mathbf{p} = \hbar\mathbf{k}$

    Superfluid if $\Delta E - \mathbf{v} \cdot \Delta\mathbf{p} > 0$ for **all** excitations
  - ideal Bose gas: $\Delta E = \mathbf{p}^2/2m \Rightarrow$ not superfluid; but interacting Bose fluid: $\Delta E \propto |\mathbf{p}|$ (sound modes) $\Rightarrow$ superfluid
  - Fermi gas: particle-hole excitations $\Delta E = 0$ possible for $|\mathbf{p}| < 2\hbar k_{\mathrm{F}} \Rightarrow$ not superfluid; but BCS state has energy gap: $\Delta E(\mathbf{p} \to 0) = \Delta \Rightarrow$ superfluid

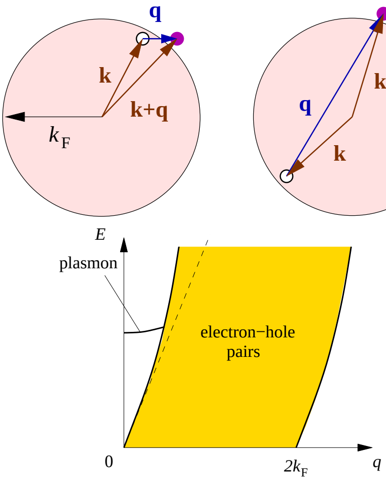
Landau criterion: Bose systems

Superfluidity requires the excited states of the fluid remain higher in energy than the ground state after the Galilean transformation to the pipe frame: $\Delta E \to \Delta E - \mathbf{v} \cdot \Delta \mathbf{p}$



You should be able to state and apply the criterion.

Landau criterion: Fermi gas



Particle-hole excitations. The net momentum $\hbar\mathbf{q}$ can be small (left) or of the order of k_F (right).

The shaded region represents the possible energies and momenta of electron-hole pairs. The dotted line has slope $\hbar v_F$.

Order parameter/condensate wavefunction

- Complex order parameter $\psi_c = \sqrt{n_s} e^{i\theta}$
  - condensate wavefunction, *e.g.* totally condensed state used in variational treatment at $T = 0 \rightarrow$ non-linear Schrödinger equation
  - more phenomenologically: $\psi_c(T)$ as an order parameter $|\psi_c|$ non-zero below T_c

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  - phase twist in space $\Rightarrow$ superfluid flow

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  - **neutral** superfluids:
    superfluid velocity $\mathbf{v}_s = \hbar\boldsymbol{\nabla}\theta/m$
    number current density $\mathbf{J} = n_s\mathbf{v}_s$ ($n_s =$ *total* density at $T = 0$)

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  - **charged** superfluids: $\mathbf{v}_s = (\hbar \boldsymbol{\nabla} \theta - 2e\mathbf{A})/m^*$
    electric current density $\mathbf{J}_e = n_s e \mathbf{v}_s$ ($n_s =$ superconducting electron density)

should be able to derive velocity from momentum operator

Characteristic lengths

- Neutral superfluids: kinetic energy *vs* interaction energy
  Non-linear Schrödinger equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi_c + \left(u|\psi_c|^2 - \mu\right)\psi_c = 0 \qquad \text{with} \quad \mu = u\bar{n}$$

  - interaction with other bosons gives potential energy: $u|\psi_c|^2$

  $\longrightarrow$ characteristic length = healing length $\xi = \hbar/(mu\bar{n})^{1/2}$
  - interaction dominates at length scales $> \xi$, *e.g.* sound wave spectrum
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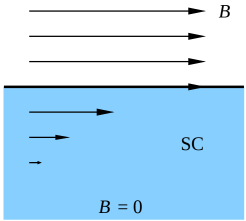
  - low-energy configurations are smooth on the scale of ξ
- Charged superfluids
  - analogous characteristic length: coherence length $\xi \simeq$ size of Cooper pairs
  - *extra* magnetic length scale: London penetration depth $\lambda_\mathrm{L} = (m_e/n_s e^2 \mu_0)^{1/2}$ (n_s = electron density) $\rightarrow$ **Meissner effect**

Superconductors: Meissner Effect

- London theory ($|\psi_c| \simeq$ constant except near wall/vortices)
  - (free) energy density in terms of magnetic field only:

$$F_{\mathrm{London}}[\mathbf{B}] = \frac{1}{2\mu_0} \int d^3\mathbf{r} \left[\lambda_{\mathrm{L}}^2 \left(\boldsymbol{\nabla} \times \mathbf{B}\right)^2 + \mathbf{B}^2\right]$$

  - Meissner effect: no fields or currents in bulk beyond London penetration depth from surface: $\lambda_{\mathrm{L}} = (m_e/n_s e^2 \mu_0)^{1/2}$
  - London equation minimises energy: $\lambda_{\mathrm{L}}^2 \nabla^2 \mathbf{B} = \mathbf{B}$



Superconductor expels weak magnetic fields.

Should be able to derive

Vortex/Flux lines

- Neutral superfluids under rotation: circulation $K = \oint_C \mathbf{v}_s \cdot d\mathbf{r}$ quantised in units of h/m
  - macroscopic angular momentum quantisation
  - vortex core is normal: size = healing length ξ
  - velocity away from core $\propto 1/r \Rightarrow$ energy $\propto \ln(R/\xi)$ ($R =$ system size)
  - rotating superfluid: first vortex line penetrates when $\omega > \omega_{c1} \propto R^{-2} \ln(R/\xi) \rightarrow 0$ for macroscopic container

Should be able to derive quantisation and compare the two cases.

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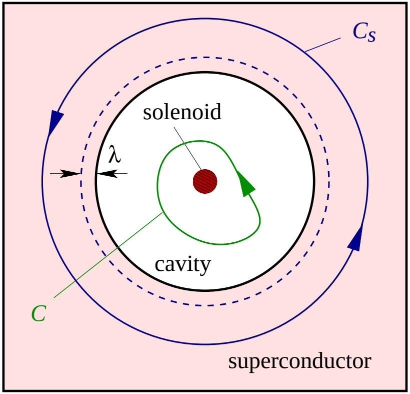
- Charged superfluids in magnetic fields: magnetic flux Φ through superconductor quantised in units of $\Phi_0 = h/2e$

$\Rightarrow$ flux lines for type II superconductors ($\lambda_{\rm L} > \xi$)
  - flux line core is normal: size = coherence length ξ
  - screening: velocity away from core $\propto e^{-r/\lambda_{\rm L}} \Rightarrow$ energy $\propto \ln(\lambda_{\rm L}/\xi)$
  - finite magnetic field needed to first flux line to penetrate: $B_{c1} \simeq \Phi_0/4\pi\lambda_{\rm L}^2$

Should be able to derive quantisation and compare the two cases.

Superconductor: flux quantisation



Superconductor with cavity
A long thin solenoid introduces magnetic flux through the cavity. Screening currents are set up at the surface of the cavity (in the ring within the dashed line).
Flux enclosed by C_s is quantised in units of $h/2e$.

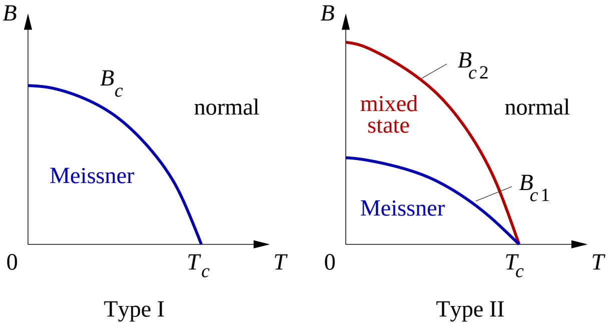
Flux line through superconductor $\simeq$ shrink cavity radius to coherence length

Should be able to derive and discuss.

Superconductor: phase diagrams

Ginzburg parameter $\kappa = \lambda_L/\xi$: $< 1/\sqrt{2} \Rightarrow$ type I, otherwise type II

Phase diagram of applied field B *vs* temperature T:



mixed state = triangular flux line lattice

Qualitative discussion for critical fields: no need for derivation.

Dynamics of a neutral superfluid

- Dynamics: collective modes in a neutral superfluid
  - density fluctuations → velocity oscillations = phase oscillations
  - Gross-Pitaevskii equation
  - normal modes $\omega_{\mathbf{k}} \sim c_s k$ for $k\xi \ll 1$

Should be able to derive from GP equation and discuss symmetry

Dynamics of a neutral superfluid

- Dynamics: collective modes in a neutral superfluid
  - density fluctuations $\longrightarrow$ velocity oscillations $=$ phase oscillations
  - Gross-Pitaevskii equation
  - normal modes $\omega_{\mathbf{k}} \sim c_s k$ for $k\xi \ll 1$
- Symmetry considerations
  - $\omega_{\mathbf{k}} \to 0$ as $k \to 0$: no restoring force for global phase shift/global translation of fluid
  - similarly with *longitudinal vibrations in solids*: no energy cost for global translation of solid

$\longrightarrow$ after second quantisation: no energy gaps $\hbar\omega_{\mathbf{k}}$ for excitations (phonons)

Should be able to derive from GP equation and discuss symmetry

Phenomenology: other topics

- Flux line motion
  - convection: vortex/flux lines flow with the local superflow $\mathbf{v}_s$
  - flux lines see Lorentz force and drag force for relative motion with superflow

$$\mathbf{f}_{\text{Lorentz}} = n_s e \Phi_0 (\mathbf{v}_s - \mathbf{v}_L) \times \hat{\mathbf{n}}$$
$$\mathbf{f}_{\text{drag}} = \eta (\mathbf{v}_s - \mathbf{v}_L) \times \hat{\mathbf{n}}$$

- Josephson effect: superconducting weak links/tunnel junctions
  - DC effect: finite current at zero voltage if there is applied flux
  - AC effect: oscillating voltage with DC current ($\leftrightarrow$ phase slip picture)

Second quantisation

- Collective excitations: from waves to particles
  - classical dynamics: small fluctuations around ground state
    $=$ superpositions of independent normal modes, frequencies $\omega_{\mathbf{k}}$
    e.g. one-dimensional chain
  - quantum mechanics:
    normal modes $\rightarrow$ quantised bosonic excitations with energy $\hbar\omega_{\mathbf{k}}$
    e.g. phonons for 1D chain
  - zero-point fluctuations $(T = 0)$
- Creation and annihilation operators for boson: $\hat{c}^{\dagger}_{\mathbf{k}}$, $\hat{c}_{\mathbf{k}}$
  - basic commutator: $[\,\hat{c}_{\mathbf{k}}, \hat{c}^{\dagger}_{\mathbf{q}}\,] = \delta_{\mathbf{k},\mathbf{q}}, \quad [\,\hat{c}_{\mathbf{k}}, \hat{c}_{\mathbf{q}}\,] = [\,\hat{c}^{\dagger}_{\mathbf{k}}, \hat{c}^{\dagger}_{\mathbf{q}}\,] = 0$
  - number operator $= \hat{c}^{\dagger}_{\mathbf{k}}\hat{c}_{\mathbf{k}}$ for bosons with wavevector $\mathbf{k}$
    Should be able to derive basic properties and calculate expectation values at $T = 0$.

Quantum Crystal

- Collective excitations: atomic displacements
  - harmonic Hamiltonian for small oscillations of a 1D chain
  - oscillations = superposition of Fourier modes
  - choice of wavevectors according to boundary conditions PS6:Q2
  - harmonic Hamiltonian $\hat{H} = \sum_{\mathbf{k}}{}' \hbar\omega_k(\hat{a}_k^\dagger \hat{a}_k + 1/2)$

Should be able to derive final form of Hamiltonian

- Quantum dynamics
  - independent excited states with energy $\hbar\omega_k \propto \sin(ka/2)$
  - rms displacement diverges with L in 1D

Should be able to calculate rms displacement

Bose fluid: second quantisation

- Collective excitations: density and phase fluctuations
  - for small density and phase fluctuations:

$$\hat{H} \simeq \int d^3\mathbf{r} \left[\frac{\hbar^2 \bar{n}}{2m} \left(|\boldsymbol{\nabla}\hat{\theta}(\mathbf{r})|^2 + \frac{1}{4}|\boldsymbol{\nabla}\hat{\nu}(\mathbf{r})|^2 \right) + \frac{u\bar{n}^2}{2}\hat{\nu}(\mathbf{r})^2 \right]$$

  $\nu(\mathbf{r})$ = fractional density change $\delta n(\mathbf{r})/\bar{n}$

  - in terms of Fourier modes: $\hat{H} = \bar{n}\sum_{\mathbf{k}} \left[A_{\mathbf{k}}\,\hat{\theta}_{\mathbf{k}}\hat{\theta}_{-\mathbf{k}} + \frac{B_{\mathbf{k}}}{4}\,\hat{\nu}_{\mathbf{k}}\hat{\nu}_{-\mathbf{k}} \right]$

Should be able to derive Fourier form of $\hat{H}$, PS7:Q3

- Second quantisation of density-phase oscillations
  - excitation spectrum $E_{\mathbf{k}} = [\epsilon_{\mathbf{k}}(2u\bar{n} + \epsilon_{\mathbf{k}})]^{1/2}$ $(\epsilon_{\mathbf{k}} = \hbar^2 k^2/2m)$
  - gapless excitations: $E_{\mathbf{k}} \to 0$ as $k \to 0$, no restoring force for global phase shift/global translation of fluid PS7:Q4

Should be able to derive excitation energies and discuss symmetry

Bose fluid: zero-point fluctuations

- Zero-point fluctuations
  - operators for density and phase fluctuations: $[\,\delta n_{\mathbf{k}}, \theta_{\mathbf{k}}\,] = i$

  $\Rightarrow$ Heisenberg uncertainty: $\Delta n_{\mathbf{k}} \Delta\theta_{\mathbf{k}} \geq 1/2$
  - number-phase uncertainty $\Rightarrow$ zero-point fluctuations: phase fluctuations are large: $(\Delta\theta_{\mathbf{k}})^2 = \langle 0|\hat{\theta}_{-\mathbf{k}}\hat{\theta}_{\mathbf{k}}|0\rangle = \epsilon_{\mathbf{k}}/4\bar{n}E_{\mathbf{k}} \sim 1/k$ for $k\xi \ll 1$
  - condensate depletion even at $T = 0$: $\langle \hat{c}^\dagger_{\mathbf{k}} \hat{c}_{\mathbf{k}}\rangle$ also $\propto 1/k$ at small k Fraction out of condensate $\propto 1/\bar{n}\xi^D$ for D dimensions
  - for fixed interaction strength u, depletion is small at a dilute 3D Bose fluid.

Many-electron systems

- Fermion creation and annihilation operators
  - antisymmetry under exchange $\Rightarrow$ anticommutation relations
    $\{\hat{c}_{\mathbf{k}}, \hat{c}^{\dagger}_{\mathbf{q}}\} = \delta_{\mathbf{k},\mathbf{q}}, \quad \{\hat{c}^{\dagger}_{\mathbf{k}}, \hat{c}^{\dagger}_{\mathbf{q}}\} = \{\hat{c}_{\mathbf{k}}, \hat{c}_{\mathbf{q}}\} = 0$
  - number operator $\hat{c}^{\dagger}_{\mathbf{k}}\hat{c}_{\mathbf{k}}$: eigenvalues 0 or 1, *i.e.* Pauli exclusion
- Fermi liquid
  - Screening between electrons: electrons dressed by positive cloud
    $\Rightarrow$ net charge for distant electrons $\simeq 0$
  - Coulomb interactions screened: Fermi liquid of weakly-interacting fermionic "quasiparticles" (effective theory for low-energy excitations)
  - Fermi surface survives moderate interactions

Superconductors: BCS theory

- Electron-phonon interaction ⇒ retarded effective electron attraction
  - electron leaves temporary “footprint” of positive ionic distortion, attracting other electrons
  - energy scale $= \hbar\omega_{\mathrm{D}}$ ($\omega_{\mathrm{D}} =$ typical phonon energies) $\ll E_{\mathrm{F}}$

  ⇒ affects thin shell of electrons with energies within $\hbar\omega_{\mathrm{D}}$ from Fermi surface

Qualitative discussion only

- Cooper instability: 2 electrons near Fermi surface PS8:Q4
  - pair wavefunction $\phi(\mathbf{r}_1 - \mathbf{r}_2)$ using states less than $\hbar\omega_{\mathrm{D}}$ above Fermi surface; spin $S = 0$
  - $\lambda_{\mathrm{BCS}} = \tilde{U} g(E_{\mathrm{F}})/V$ is dimensionless measure of attractive strength
  - Schrödinger equation ⇒ binding energy $= 2\hbar\omega_{\mathrm{D}}\, e^{-2/\lambda_{\mathrm{BCS}}}$
  - pairing for arbitrarily weak attraction

Should be able to derive Cooper instability

Superconductors: BCS ground state

- BCS wavefunction $\prod_{\mathbf{k}} \left(u_{\mathbf{k}} + v_{\mathbf{k}} \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} \right) |\text{vac}\rangle$ with $u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1$
  - Fermi gas: $u_{\mathbf{k}} = 0, v_{\mathbf{k}} = 1$ for $k < k_{\text{F}}$ and $u_{\mathbf{k}} = 1, v_{\mathbf{k}} = 0$ for $k > k_{\text{F}}$
  - does not have a definite particle number, but *fractional* number variation is small: $\Delta N/N \sim 1/N^{1/2}$
- Order parameter = BCS gap:
  $\Delta = \frac{\tilde{U}}{V} {\sum_{\mathbf{k}}}' \langle \hat{c}^{\dagger}_{\mathbf{k}\uparrow} \hat{c}^{\dagger}_{-\mathbf{k}\downarrow} \rangle = \frac{\tilde{U}}{V} \sum_{\mathbf{k}} u_{\mathbf{k}} v_{\mathbf{k}}$
  - Variational theory:
    $$1 = \frac{\tilde{U}}{2V} {\sum_{\mathbf{k}}}' \frac{1}{(\Delta^2 + \epsilon_{\mathbf{k}}^2)^{1/2}} \quad \Rightarrow \quad \Delta = 2\hbar\omega_{\text{D}} e^{-1/\lambda_{\text{BCS}}}$$
    ($\epsilon_{\mathbf{k}}$: energy measured from E_{F})
  - $|\Delta|$ also energy gap for excitations: $E_{\mathbf{k}} = (\Delta^2 + \epsilon_{\mathbf{k}}^2)^{1/2}$
  - coherence length $\xi = \hbar v_{\text{F}}/\Delta \simeq$ Cooper pair size

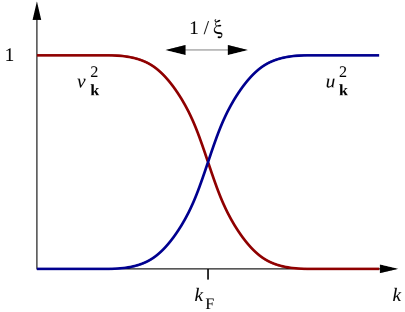
Understand derivations

Superconductors: BCS coherence factors

Deviate from non-interacting values near k_F: width

$$\Delta k = \Delta/\hbar v_F = 1/\xi$$

$\Rightarrow$ pairing amplitude $\langle \hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow}\rangle = u_\mathbf{k} v_\mathbf{k}$ non-zero in range $\pm\Delta k$ near k_F



Superconductors: other topics

- Pairing symmetries
  - $\Delta_{\mathbf{k}}$ can depends on $\mathbf{k}$
  - even angular momentum: s, d-wave requires $S = 0$
  - odd angular momentum: p-wave requires $S = 1$
- Excitations
  - nature of excitations: neither particle nor hole...
  - probing excitations: EM absorption, photoemission, ...

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