Additional exercises

Statistical Mechanics 2006–2007

Prof Kim Christensen
Blackett Laboratory, Room 816A
Imperial College London
Prince Consort Road
SW7 2BW London
United Kingdom

October 9, 2006

1. *Taylor expansion*

The *Taylor expansion* of a function $f(x)$ about a point $x = x_0$ to order n is given by

$$\begin{aligned} f(x) &\approx f(x_0) + f^{(1)}(x_0)(x - x_0) + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\ &= \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j, \end{aligned} \tag{1.1}$$

where $f^{(j)}(x_0)$ is the jth derivative of the function $f(x)$ evaluated at $x = x_0$ with the convention that the 0th derivative $f^{(0)}(x_0) = f(x_0)$. Note that the factorial in the denominator ensures that the jth derivative of the Taylor expansion evaluated in $x = x_0$ equals $f^{(j)}(x_0)$. Therefore, the Taylor expansion Equation (1.1) is the polynomial of order n that best approximates the function $f(x)$ about the point $x = x_0$. If $x_0 = 0$, that Taylor expansion to order n is also known as the Maclaurin expansion to order n.

(i) Find the Taylor expansion of the function $f(x) = \ln(1 - x)$ about the point $x = 0$ to order 3 and show that

$$\ln(1 - x) \to -x \quad \text{for } x \to 0. \tag{1.2}$$

(ii) Do a Taylor expansion of the function

$$f(p) = p\left[1 - \left(\frac{1-p}{p}\right)^3\right] \quad \text{for } p > 0 \tag{1.3}$$

about the point $p = 1/2$ to show that

$$f(p) \approx B(p - 1/2) + C(p - 1/2)^2 \quad \text{for } p \to 1/2 \tag{1.4}$$

and identify the coefficients B and C.

(iii) Plot the function $f(p)$ and its Taylor expansion to second order versus $(p - 1/2)$ for $p > 1/2$ on the same double-logarithmic graph. In which range of $p - 1/2$ does the Taylor expansion to second order well approximate the function $f(p)$?

2. *Power-law probability density*

You have been hired to build and manage a dam to prevent a river from overflowing. A natural spring guarantees that the level h of the river is always positive, i.e., $h \geq h_{min} > 0$. Measurements of the level of the river once a day show that it is described by a power-law probability density,

$$P(h) = \begin{cases} Ah^{-2} & \text{for } h \geq h_{min} \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

where A is a constant.

(i) Show that the probability density is normalised if $A = h_{min}$.

(ii) (a) What is the probability, on a given day, of measuring a level $h \geq h_{max}$ with $h_{max} \geq h_{min}$?

(b) On average, how many days would you have to wait to see one event with $h \geq h_{max}$?

(iii) Your job contract has a clause that if the dam overflows, you get fired.

(a) Can you build a dam that would guarantee your job forever? Explain your answer.

(b) You would like to keep your job for N years with probability p. How high should you build the dam?

(c) Find the height of the dam, assuming you want to keep your job for 10 years with 90% probability when $h_{min} = 0.01$ m.

(iv) (a) Find the average height $\langle h \rangle$ of the river.

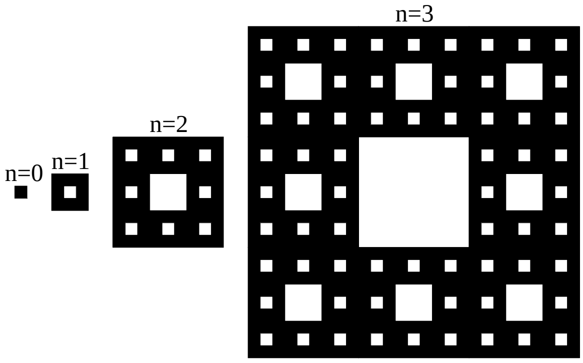
(b) Discuss modification(s) to the probability density $P(h)$ that would make the model more physically realistic.

3. *Density of fractal object*

The mass M of a fractal geometrical object grows with the linear size ℓ of the object as

$$M(\ell) = \ell^D.$$

The exponent D is the fractal dimension of the object.



Iteration	n = 0	n = 1	n = 2	n = 3
Linear size ℓ	1	3	9	27
No. black unit squares	1	8	64	512
No. white unit squares	0	1	17	217

(i) Consider $d = 2$ and the object displayed above. At each iteration n, the linear size ℓ is enlarged by a factor 3 and a black unit square is replaced with 3×3 unit squares in which the centre square is white, while a white unit square is replaced with 3×3 white unit squares. Assume the mass of a unit square (black or white) is 1.

(a) Express the mass M and the linear size ℓ of the geometrical object of black squares as a function of the number of iterations n and show that $D_{black} \approx 1.89 < d$.

(b) Find the density $\rho(\ell)$ of the fractal object of black squares as a function of ℓ. What is the limit $\lim_{\ell \to \infty} \rho(\ell)$?

(c) Show that, in the limit of $\ell \to \infty$, the geometrical object of white squares has $D_{white} = 2$.

(ii) Oil resides in connected pores inside porous material in $d = 3$. Samples of volumes $0.001\,\mathrm{m}^3$, $0.008\,\mathrm{m}^3$ and $0.064\,\mathrm{m}^3$ from an oil field have densities of oil $250\ \mathrm{kg\,m^{-3}}$, $177\,\mathrm{kg\,m^{-3}}$ and $125\,\mathrm{kg\,m^{-3}}$, respectively.

(a) Demonstrate that this is consistent with the oil residing on a fractal and determine its fractal dimension D_{oil}.

(b) How much oil can be recovered from an oil field of volume $10^3\mathrm{km}^3$?

4. *Correlation function and critical exponent ν in $d = 1$ and a sum rule*
Consider site percolation in $d = 1$. Define the site-site correlation function $g(\mathbf{r}_i, \mathbf{r}_j)$ as the probability that a site at position $\mathbf{r}_j$ from an occupied site at position $\mathbf{r}_i$ belongs to the same finite cluster.

(i) Show that

$$g(\mathbf{r}_i, \mathbf{r}_j) = \exp(-|\mathbf{r}_i - \mathbf{r}_j|/\xi) \tag{4.1}$$

with an appropriate definition of the correlation length ξ.

(ii) Generally, we define the critical exponent ν by

$$\xi(p) \propto |p - p_c|^{-\nu} \quad \text{for } p \to p_c. \tag{4.2}$$

Show that for $d = 1$ percolation, $\nu = 1$.

(iii) The average cluster size $\chi(p) = \frac{\sum_{s=1}^{\infty} s^2 n(s,p)}{\sum_{s=1}^{\infty} sn(s,p)} = \frac{1+p}{1-p}$ for $0 < p < 1$. Show that the sum over all positions $\mathbf{r}_j$ of the site-site correlation function

$$\sum_{\mathbf{r}_j} g(\mathbf{r}_i, \mathbf{r}_j) = \chi(p). \tag{4.3}$$

Hint: Remember to include the neighbours to the left and right of the occupied site at $\mathbf{r}_i$.

(iv) Although the site-site correlation function is not given by Equation (4.1) in dimensions $d > 1$, the sum rule in Equation (4.3) does remain valid for $d > 1$. Explain in words why that is.
Hint: Refer to the definition of the correlation function.

5. *Data collapse for the cluster number density in $d = 1$.*

The cluster number density in $d = 1$ is

$$n(s,p) = (1-p)^2 p^s = (1-p)^2 \exp(-s/s_\xi) \tag{5.1}$$

with the characteristic cluster size

$$s_\xi(p) = -\frac{1}{\ln(p)}. \tag{5.2}$$

(i) In each of the three following sub-questions, plot the given quantities on the same Figure for various values of the occupation probabilities, say $p = 0.905, 0.99, 0.999, 0.9999$.

(a) Figure (a): plot $\log[n(s,p)]$ versus $\log(s)$ for the four values of p.

(b) Figure (b): plot $\log[s^2 n(s,p)]$ versus $\log(s)$ for the four values of p.

(c) Figure (c): plot $\log[s^2 n(s,p)]$ versus $\log(s/s_\xi)$ for the four values of p.

(ii) Consider the following re-formulation of the cluster number density in $d = 1$

$$\begin{aligned} n(s,p) &= (1-p)^2 \exp(-s/s_\xi) \\ &= (p_c - p)^2 \exp(-s/s_\xi) \quad \text{since } p_c = 1 \\ &= s^{-2}[s(p_c - p)]^2 \exp(-s/s_\xi) \\ &\approx s^{-2}(s/s_\xi)^2 \exp(-s/s_\xi) \quad \text{since } s_\xi(p) \approx (p_c - p)^{-1} \quad \text{for } p \to p_c^- \\ &= s^{-2}\mathcal{G}_{1d}(s/s_\xi) \end{aligned} \tag{5.3}$$

where the function

$$\mathcal{G}_{1d}(x) = x^2 \exp(-x). \tag{5.4}$$

Relate the equation

$$n(s,p) = s^{-2}\mathcal{G}_{1d}(s/s_\xi) \tag{5.5}$$

to your plot in Figure (c).

6. *System of non-interacting spins.*

(i) To gain experience in statistical mechanical calculations, work your way through Section 2.2 (pages 124 – 131) in the book *Complexity and Criticality* for a system of N non-interacting spins.

(ii) Find the total number of microstates. How many $\Omega(n)$ of these microstates has n spins orientated parallel to the external field H?

(iii) In the ground state, all the spins are aligned with the external field, and $\Omega(N) = 1$. In a first exited state, one spin is flipped, and $\Omega(N-1) = N$. For which n is $\Omega(n)$ maximal? What is $\Omega(0)$? Sketch $\Omega(n)$ as a function of n.

7. *Widom scaling and finite-size scaling.*

Define the reduced (dimensionless) temperature, t, as a measure of the deviation from the critical temperature T_c and reduced (dimensionless) external field, h, by

$$t = \frac{T - T_c}{T_c} \tag{7.1a}$$
$$h = \frac{H}{k_B T}. \tag{7.1b}$$

In a real-space renormalisation transformation, all spins s_i in a block I of volume b^d is replaced by a single block spin s_I and all length scales are rescaled by the factor b. Assume that the associated rescaled temperature and external field

$$t' = tb^{y_T} \qquad y_T > 0 \tag{7.2a}$$
$$h' = hb^{y_H} \qquad y_H > 0. \tag{7.2b}$$

It can be shown that the partition function $Z(N', t', h') = Z(N, t, h)$ implying that the free energy per spin is a homogeneous function, that is,

$$f_s(t, h) = b^{-d} f_s(tb^{y_T}, hb^{y_H}) \quad \forall b, y_T > 0, y_H > 0. \tag{7.3}$$

(i) Show that

$$y_T = 1/\nu. \tag{7.4}$$

(ii) Show that Equation (7.3) implies the the Widom scaling form of the singular part of the free energy per spin

$$f_s(t, h) = |t|^{2-\alpha} \mathcal{F}_\pm \left(h/|t|^\Delta\right) \quad \text{for } t \to 0^\pm \text{ and } |h| \ll 1. \tag{7.5}$$

with

$$d\nu = 2 - \alpha \tag{7.6a}$$
$$y_H / y_T = \Delta. \tag{7.6b}$$

(iii) Using Equation (7.5), find the scaling form of the heat capacity per spin.

Equation (7.3) may be generalised to a finite system of linear size L where

$$f_s(t, h, L) = b^{-d} f_s(tb^{y_T}, hb^{y_H}, Lb^{-1}) \quad \forall b, y_T > 0, y_H > 0. \tag{7.7}$$

(iv) Show that the susceptibility per spin in zero external field

$$\chi(t, L) \propto |t|^{-\gamma} \chi_\pm(L/\xi). \tag{7.8}$$

(v) From your experience with finite-size scaling in percolation, find

$$\chi(t, L) = \begin{cases} ? & \text{for } L \gg \xi \\ ? & \text{for } L \ll \xi \end{cases} \tag{7.9}$$

Explain how you might use this result to measure numerically the critical exponents.

CONDENSED MATTER THEORY MOCK EXAM MAY 2001

The paper consists of *three* sections: A & B & C.
Section A contains *two* questions.
Section B contains *two* questions.
Section C contains *two* questions.
All questions carry equal marks.

Answer THREE questions, taking ONE question from section A, ONE question from section B, and ONE question from section C.

SECTION A

1. (i) Consider percolation on an infinite $d = 2$ square lattice at the critical occupation probability $p = p_c$. How does the mass $M_\infty(p_c, \ell)$ of the percolating (infinite) cluster within a square of linear size ℓ depend on ℓ? How does the corresponding density of occupied sites $P_\infty(p_c, \ell)$ belonging to the infinite cluster within a square of linear size ℓ depend on ℓ? What is $\lim_{\ell\to\infty} P_\infty(p_c, \ell)$?

(ii) Consider a real space renormalisation of the lattice by a factor b, that is, $\ell \to \ell/b$. Justify the relation

$$M_\infty(p_c, \ell) = b^D M_\infty(p_c, \ell/b).$$

(iii) Assume that the occupation probability $p > p_c$ and let ξ denote the correlation length. Let $M_\infty(\xi, \ell)$ denote the mass of the infinite cluster within a square of linear size ℓ. Discuss $M_\infty(\xi, \ell)$ in the limits $\ell \ll \xi$ and $\ell \gg \xi$. Write a scaling form for $M_\infty(\xi, \ell)$ and identify the crossover function.

(iv) Figure 1 below (double-logarithmic plot) displays the measured density $P_\infty(p, \ell)$ of sites belonging to the infinite cluster within a square of linear size ℓ at two different values of $p > p_c$ in a finite lattice. The dashed line has slope -0.11. Discuss the behaviour of the graphs as a function of ℓ.

(v) Derive the hyper scaling relation $D = d - \beta/\nu$.

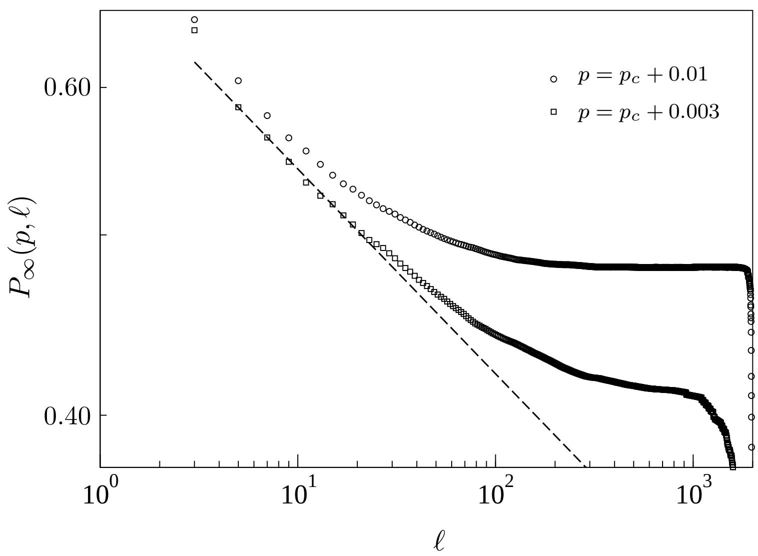


Figure 1: The density of the percolating cluster, $P_\infty(p, \ell)$, as a function of window size ℓ for two different occupation probabilities $p > p_c$ in a system of size $L = 2000$.

2. *Bond percolation and real space renormalisation for a square lattice*
Site percolation has a counterpart called bond percolation. Each bond between neighbouring lattice sites can be occupied (open) with probability p and empty (closed) with probability $(1-p)$. A cluster is a group of connected occupied (open) bonds. The percolation threshold for a square lattice is $p_c = 0.5$.

In a bond percolation real space renormalisation on the square lattice, the lattice is replaced by a new lattice, with bond length b, as shown by the dashed lines in the figure below.

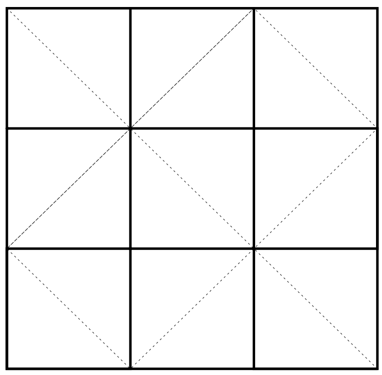
(i) Show that the probability to have a spanning (percolating) cluster diagonally in a unit square is $2p^2 - p^4$.

(ii) Identify the fixed points of the renormalisation transformation

$$R_b(p) = 2p^2 - p^4,$$

and comment on their nature.

(iii) Explain how the critical exponent ν is related to the renormalisation transformation $R_b(p)$ and find the value of ν.



SECTION B

3. A system consists of N non-interacting spins $s_i, i = 1, \ldots, N$ in an external field H. Each spin is either parallel or anti-parallel to H.

(i) Justify that the total energy

$$E = -\sum_{i=1}^{N} s_i H,$$

with $s_i = +1$ for a parallel spin and $s_i = -1$ for an anti-parallel spin.

(ii) Show that, at temperature T, the partition function of the system $Z(T, H) = [2\cosh(\beta H)]^N$.

(iii) Calculate as functions of temperature T and the external field H the free energy F, the entropy S, and the average energy $\langle E \rangle$.

(iv) Calculate the mean magnetisation of the i'th spin $\langle s_i \rangle$.

(v) Define and calculate the susceptibility χ. Why is the susceptibility also known as a *response function*?

4. (i) Starting from the Ising model energy

$$E = -J \sum_{\langle i,j \rangle} s_i s_j - H \sum_{i=1}^{N} s_i$$

show that the mean-field energy can be written on the form

$$E = -\left(Jqm + H\right) \sum_{i=1}^{N} s_i,$$

where q is the coordination number and $m = \langle s_j \rangle$.

(ii) In a system with N spins, show that the partition function

$$Z = \left[2 \cosh \left(\beta J q m + \beta H\right)\right]^N$$

(iii) Find the free energy per spin $f(T, H) = \frac{F}{N}$ and comment upon the result.

(iv) Find the magnetisation per spin $m(T, H)$ and show that

$$m_0(T) = \lim_{H \to 0^{\pm}} m(T, H) = \tanh\left(\beta J q m_0(T)\right).$$

(v) Show that a critical temperature T_c exists and identify T_c.

SECTION C

5. (i) Given a model system displaying self-organised criticality with an event size probability

$$P(s) = \begin{cases} s^{-\tau} & \text{for } s \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Derive an inequality for τ such that $P(s)$ can be normalised. Derive an inequality for τ such the the average avalanche size $\langle s \rangle = \sum_{s=1}^{\infty} sP(s)$ diverges.

(ii) Consider the model system where the level of conservation is defined by a parameter α and where the event size probability is given by

$$P(s) \propto \begin{cases} s^{-\tau} H(1 - s/s_\xi) & \text{for } s \geq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where the *Heaviside* function

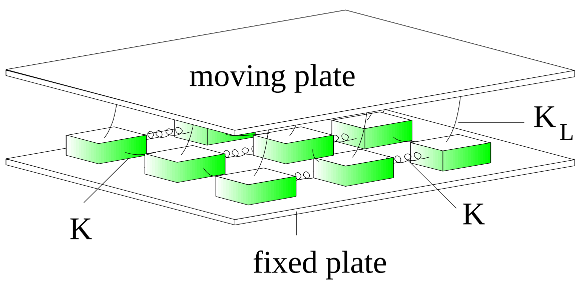
$$H(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and the characteristic event size $s_\xi \propto (\alpha_c - \alpha)^{-1/\sigma}$ for $\alpha < \alpha_c$. Make a sketch of $P(s)$ for three different level of conservation $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_c$. What is the limiting function of $P(s)$ for $\alpha \to \alpha_c$?

(iii) Assume the scaling form of $P(s)$ given in (ii). Derive a scaling relation between the scaling exponent γ_k describing the divergence of the k'th moment $\langle s^k \rangle = \sum_{s=1}^{\infty} s^k P(s)$ when $\alpha \to \alpha_c$ and the critical exponents τ and σ.

(iv) Define explicitly the dynamical rules for the Olami-Feder-Christensen spring-block model on a $d = 2$ square lattice in terms of the force F_i on block i and the parameter $\alpha = K/(4K + K_L)$, see figure below. (There is **no** need to derive the rules). Discuss briefly: *the time scale of drive/response*, and *the level of conservation*.

(v) Define the concepts of the OFC model being (a) critical and (b) noncritical. How does the model behave for $\alpha = 0$? How would you expect the model to behave for $\alpha = 0.25$? Discuss briefly the possibility for a phase transition, that is, the existence of a critical value of α_c above which the model is critical and below which the model is noncritical.



6. Not relevant for Statistical Mechanics 2006/2007

Define and discuss briefly the concepts of *critical external magnetic field* $H_c(T)$, type I superconductors, and type II superconductors.

(ii) Assume that an electric field $\mathbf{E}$ arises within a superconductor. Let $\mathbf{j}$ denote the current density of the superconducting electrons. Using a free electron dynamics approach, show that

$$\frac{d\mathbf{j}}{dt} = \frac{n_s e^2}{m}\mathbf{E},$$

where n_s denotes the density of superconducting electrons.

Hence show that

$$\frac{\partial}{\partial t}\left(\nabla \times \mathbf{j} + \frac{n_s e^2}{m}\mathbf{B}\right) = 0. \qquad (6.1)$$

(iii) Discuss whether Equation (6.1) together with the Maxwell equations

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

imply $\mathbf{B} = \mathbf{0}$ inside a type I superconductor under the conditions $T < T_c$ and $H < H_c(T)$.

(iv) The phenomenological London equation is

$$\nabla \times \mathbf{j} = -\frac{1}{\mu_0 \lambda_L^2}\nabla \times \mathbf{A} = -\frac{1}{\mu_0 \lambda_L^2}\mathbf{B}.$$

Discuss the relationship of the London equation with Equation (6.1). Discuss whether the London equation imply $\mathbf{B} = \mathbf{0}$ inside a type I superconductor under the conditions $T < T_c$ and $H < H_c(T)$.

Show that the London equation imply that the current in a superconductor can exist only within a layer of thickness λ_L.

(v) Consider a superconducting ring. Define the phenomenon of flux quantisation. Identify the terms in the equation for the electrical current density

$$\mathbf{j} = q\,\mathbf{j}_{probability} = \frac{q}{m}[\hbar\nabla\Theta(\mathbf{r}) - q\mathbf{A}]n_s$$

and show that the equation implies the London equation as well as the flux quantisation. Comment on the value of the charge q.

CONDENSED MATTER THEORY EXAM MAY 2001

The paper consists of *three* sections: A & B & C.
Section A contains *two* questions.
Section B contains *two* questions.
Section C contains *two* questions.
All questions carry equal marks.

Answer THREE questions, taking ONE question from section A, ONE question from section B, and ONE question from section C.

SECTION A

1. Consider the following percolation problem involving a cluster of a very large but finite size s at the critical occupation probability $p = p_c$. The part of the cluster contained within a box of linear size ℓ has mass $M(\ell, R_s)$, where R_s denotes the radius of gyration.

(i) Define R_s and discuss the dependence of M on ℓ and R_s. Writing

$$M(\ell, R_s) = \ell^A m(\ell/R_s^B), \tag{1.1}$$

identify the exponents A and B, and find the behaviour of the crossover function $m(x)$ for $x \ll 1$ and $x \gg 1$.

(ii) Consider a d-dimensional lattice of linear size L at the critical occupation probability $p = p_c$ and a real space renormalisation of the lattice by a factor b. Explain why a cluster containing s sites in the lattice L^d is mapped into a cluster having only $s' = s/b^D$ sites in the new lattice $(L/b)^d$, where D denotes the fractal dimension.

(iii) Let $n(s, p)$ denote the cluster number density, that is, the average number of clusters containing s sites per lattice site. Justify the relation

$$sn(s, p_c, L) = b^{-d} s' n(s', p_c, L/b), \tag{1.2}$$

with $s' = s/b^D$.

(iv) Assuming the scaling form of the cluster number density

$$n(s, p) = s^{-\tau} \mathcal{G}(s/s_\xi) \quad \text{for } p \to p_c, s \gg 1,$$

show that in a d-dimensional lattice of size L one would expect

$$n(s, p_c, L) = s^{-\tau} g(s/L^D). \tag{1.3}$$

What is the form of the scaling function $g(x)$? Explain your answer.

(v) Combine Equations (1.2) and (1.3) to show the hyper-scaling relation

$$\tau = \frac{d + D}{D}.$$

2. *Site percolation and real space renormalisation for a square lattice*

(i) Define and outline the procedure of real space renormalisation transformation applied to the percolation theory problem.

(ii) Consider site percolation on a square lattice in two dimensions. Using blocks of size 2×2 and adapting the spanning cluster (in any direction) rule to define the real space renormalisation transformation, show that

$$R_b(p) = p^4 - 4p^3 + 4p^2.$$

(iii) Find the fixed points for the real space renormalisation transformation above and comment on their nature. What are the correlation lengths ξ associated with the respective fixed points? Discuss the concept of flow in parameter space (restricted to p-space) associated with the real space renormalisation transformation R_b.

(iv) Identify the critical occupation probability p_c, derive the equation used to determine the correlation length exponent ν predicted by the real space renormalisation transformation, and evaluate ν. Compare the findings to the analytic results and comment on the discrepancies.

(v) Discuss the concept of universality in the theory of percolation. Give examples of quantities which are universal and non-universal, respectively.

SECTION B

3. (i) Write down the energy $E_{\{s_i\}}$ for a ferromagnetic spin 1/2 Ising model with coupling constant J and external field H. Discuss the approximations entering into this model.

(ii) Write down the energy E_{1d} in zero external field of a one-dimensional ($d = 1$) Ising model consisting of N spins with periodic boundary conditions. What is the ground state energy per spin E/N of the system and the associated ground states? What is the associated magnetisation per spin $m = M/N$ of the system in the ground states?

(iii) Using the 2×2 transfer matrix $\mathbf{T}$ with elements

$$T_{s_i s_{i+1}} = \exp(J s_i s_{i+1}/k_B T),$$

show that the partition function

$$Z = \mathrm{Tr}\,(\mathbf{T}^N)$$

(iv) Hence, show that the free-energy per site in the $N \to \infty$ limit is

$$f = -J - k_B T \ln[1 + \exp(-2J/k_B T)].$$

(v) Discuss the physics of the above result and comment on the possibility of a phase transition in the $d = 1$ Ising model.

4. (i) Consider a lattice where only a fraction p of the lattice sites are occupied by ferromagnetic spins which can be in one of two states $s_i = \pm 1$ and the remaining fraction $1-p$ are nonmagnetic (empty). Consider a finite percolation cluster of size s. What is the ground state of the cluster when $T \to 0$?

(ii) Let J denote the coupling constant and consider a very low temperature $k_BT \ll J$ and a small external field H. Show that the probability for a given cluster of size s to point in the direction of H is

$$p_{\uparrow\uparrow} = \frac{\exp(sH/k_BT)}{\exp(sH/k_BT) + \exp(-sH/k_BT)}$$

and that the probability for the $s-$cluster to point in the opposite direction of H is

$$p_{\uparrow\downarrow} = \frac{\exp(-sH/k_BT)}{\exp(sH/k_BT) + \exp(-sH/k_BT)}.$$

(iii) Show that the magnetisation per $s-$cluster

$$m_{s-cluster} = s\tanh(sH/k_BT).$$

Hence, deduce that the total magnetisation per lattice site is

$$m(p,H) = \pm P_\infty(p) + \sum_s sn(s,p)\tanh(sH/k_BT),$$

where $n(s,p)$ is the cluster number density at occupation probability p. What does $P_\infty(p)$ represent?

(iv) Find the magnetisation per spin $m_0(p) = \lim_{H\to 0} m(p,H)$ in zero external field for p close to p_c. Furthermore, show that for small external field $H \ll k_BT$, the susceptibility per spin

$$\frac{\chi}{N} = \left(\frac{dm}{dH}\right)_{H=0} \propto |p-p_c|^{-\gamma} \quad \text{for } p \to p_c,$$

where γ is the critical exponent from percolation.

(v) For $p < p_c$, use the general scaling ansatz for the cluster number density

$$n(s,p) = s^{-\tau}\mathcal{G}(s/s_\xi) \quad \text{for } p \to p_c, s \gg 1,$$

to show that

$$m(p,H) = (p_c - p)^\beta g(Hs_\xi) \quad \text{for } p \to p_c, s \gg 1.$$

Identify β and discuss the behaviour of $g(x)$ for small and large x.

Note: $\tanh(x) = \frac{\exp(x)-\exp(-x)}{\exp(x)+\exp(-x)} = x - \mathcal{O}(x^3)$ *for small* x.

SECTION C

5. (i) Explain the difference between criticality (scale invariance) in equilibrium systems and non-equilibrium systems displaying self-organised criticality (SOC).

(ii) Define explicitly the Bak-Tang-Wiesenfeld sandpile model on a $d = 2$ square lattice. Define and discuss briefly:

(a) the time scale of drive/response,

(b) the threshold value,

(c) avalanches,

(d) the transient period,

(e) statistically stationary states, and

(f) the attractor.

Explain why one would expect to see a broad distribution of avalanche sizes.

(iii) Given a model system of linear size L displaying self-organised criticality with an event size probability

$$P(s, L) \propto s^{-\tau} \exp(-s/L^D), \quad \text{for } s \gg 1. \tag{5.1}$$

Make a sketch of $P(s, L)$ for three different system sizes $L_1 < L_2 < L_3$. Explain how to make a data collapse and identify the scaling function f.

(iv) Derive a scaling relation between the the scaling exponent γ_k describing the divergence of the kth moment $\langle s^k \rangle = \sum_{s=1}^{\infty} s^k P(s, L)$ with system size and the critical exponents τ and D.

(v) Consider the event size probability given in Equation (5.1) Assume the scaling ansatz

$$P(s, L) \propto L^{-\beta} g(s/L^D).$$

Find a scaling relation between the critical indices β, τ, and D. What is the relation between the scaling functions g and f?

6. Not relevant for Statistical Mechanics 2006/2007

Consider the phase-transition associated with a specimen displaying superconductivity. Define/discuss the concepts of *critical temperature* T_c, *Cooper pairs* and *order parameter.*

(ii) Consider the simple Landau theory for a 2nd order phase transitions based on the expansion of the free energy in powers of the (norm of the) order parameter

$$F_S = a_0 + a_2|\Psi|^2 + a_4|\Psi|^4 + \ldots, \quad a_4 > 0,$$

where the subscript S refers to the superconducting state. Truncate the power series for the free energy F_S to include only terms up to the $4th$ order. Based on your knowledge of the phase transition in a superconductor, discuss why, for $T \to T_c$, we may write $a_2 = \tilde{a}_2(T - T_c)$, with $\tilde{a}_2 > 0$, and $a_0 = F_N$, with subscript N referring to the normal state.

(iii) Assuming zero external magnetic field, use the Landau theory to find the differences in the free energy $F_S - F_N$ and in the entropy $S_S - S_N$ for T close to T_c. Comment on the physical interpretation. Furthermore, discuss these results in the light of the fundamental equation

$$F = \langle E \rangle - TS.$$

(iv) Define the Meissner effect. Show explicitly, that the Meissner effect is not a consequence of the material being a perfect conductor.

(v) The phenomenological London equation is

$$\nabla \times \mathbf{j} = -\frac{1}{\mu_0 \lambda_L^2} \nabla \times \mathbf{A} = -\frac{1}{\mu_0 \lambda_L^2} \mathbf{B}.$$

Show that the London equation (together with Maxwell's equations) imply the Meissner effect and comment on the physical interpretation of λ_L.

Note: You may use the following differential vector identity: $\nabla \times (\nabla \times F) = \nabla(\nabla \cdot F) - \nabla^2 F$.

CONDENSED MATTER THEORY EXAM MAY 2002

The paper consists of *three* sections: A & B & C.
Section A contains *two* questions.
Section B contains *two* questions.
Section C contains *two* questions.
All questions carry equal marks.

Answer THREE questions, taking ONE question from section A, ONE question from section B, and ONE question from section C.

SECTION A

1. For percolation in a lattice of infinite size $L = \infty$, the average cluster size at occupation probability p diverges like

$$\chi(p, L = \infty) \propto |p - p_c|^{-\gamma} \quad \text{for } p \to p_c. \tag{1.1}$$

(i) Assuming the scaling form of the cluster number density

$$n(s, p) = s^{-\tau}\mathcal{G}(s/s_\xi) \quad \text{for } p \to p_c \text{ and for all cluster sizes } s, \tag{1.2}$$

where the scaling function $\mathcal{G}$ is constant for small arguments and decays rapidly for large arguments, find a scaling relation between the critical exponents γ, τ and σ.

(ii) Let $\xi(p)$ denote the correlation length at occupation probability p. Show that for p close to p_c,

$$\chi(\xi, L = \infty) \propto \xi^{\gamma/\nu}. \tag{1.3}$$

(iii) (a) Argue why, for finite lattices $L < \infty$, with $L \ll \xi$ one would expect

$$\chi(\xi, L) \propto L^{\gamma/\nu} \tag{1.4}$$

(b) For finite lattices, $L < \infty$, show that the average cluster size $\chi(\xi, L)$ obeys the scaling law

$$\chi(\xi, L) = \xi^{\gamma/\nu} f(L/\xi) \tag{1.5}$$

Identify the behaviour of the scaling function f for $L \gg \xi$ and $L \ll \xi$ and sketch $\log f(x)$ versus $\log x$.

(c) Numerically, how would you determine the ratio γ/ν?

(iv) In finite lattices, $L < \infty$, you may assume the cluster number density at $p = p_c$ obeys the scaling law

$$n(s, p_c, L) = s^{-\tau} g(s/L^D), \tag{1.6}$$

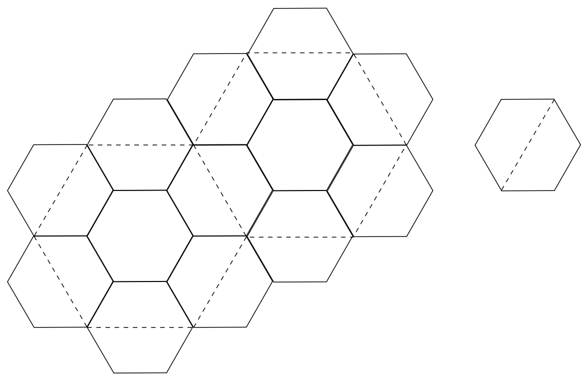
where D is the fractal dimension and $g(x)$ is constant for $x \ll 1$ and decays rapidly for $x \gg 1$.

Combining Equations (1.5) and (1.6) or otherwise, derive the scaling relation

$$\gamma/\nu = D(3 - \tau). \tag{1.7}$$

2. *Bond percolation and the real space renormalisation transformation for a honeycomb lattice*

Consider bond percolation on the honeycomb lattice in two dimensions shown below and let p denote the occupation probability. In a real space renormalisation transformation approach, the cells containing dashed lines in the figure are replaced by single bonds yielding a larger honeycomb lattice containing the dashed bonds only.



(i) (a) Show that the probability of having a cell with four bonds present with a spanning cluster along the dashed line is $6p^4(1-p)^2$.

(b) Using the spanning cluster rule (along the dashed line) to define the real space renormalisation transformation $R_b(p)$, show that

$$R_b(p) = 2p^3 - p^6, \tag{2.1}$$

and identify b, assuming the original bond length is 1.

(ii) (a) Given that $R_b(0.848) = 0.848$, sketch the real space renormalisation transformation in Equation (2.1) as a function of occupation probability p.

(b) Solve graphically the fixed point equation for the real space renormalisation transformation R_b. Why are the fixed points $p^\star$ of a real space renormalisation transformation associated with scale invariance?

(c) Identify the underlying scale invariance of the microscopic states associated with the fixed points. What are the correlation lengths in these states?

(iii) Discuss the flow in p-space associated with the real space renormalisation transformation R_b and relate it to the renormalisation of the correlation length.

(iv) Identify the critical occupation probability p_c and determine the correlation length exponent ν predicted by the real space renormalisation transformation. Compare the findings to the known numerical results of $p_c \approx 0.653$ and the exact value of ν in two dimensions. Comment on the discrepancies.

SECTION B

3. The energy $E_{\{s_i\}}$ for a ferromagnetic spin 1/2 Ising model with coupling constant $J > 0$ in zero external field is

$$E_{\{s_i\}} = -J \sum_{\langle ij \rangle} s_i s_j$$

where $s_i = \pm 1$ is the spin at lattice position i and the sum runs over different pairs of nearest neighbour sites.

(i) Consider N spins on a regular cubic lattice in d dimensions with periodic boundary conditions. At $T = 0$, what are the ground state energy E and the possible values of the magnetisation M of the Ising model.

(ii) Describe qualitatively, the changes in the microscopic states as the temperature T decreases from ∞ to 0. At which temperatures are the microscopic states self-similar and what are the associated correlation lengths? Make a sketch of the average magnetisation per spin m_0 as a function of T.

(iii) The Landau theory for the Ising model expands the free energy per spin f in terms of the order parameter m_0 as

$$f(T, m_0) = a_0 + a_2 m_0^2 + a_4 m_0^4 \tag{3.1}$$

Why are terms of odd powers excluded in the expansion Equation (3.1)? What does a_0 represent and why can we assume $a_4 > 0$? What is the temperature dependence of the term a_2? Sketch the function $f - a_0$ as a function of m_0 for (a) $T > T_c$, (b) $T = T_c$, and (c) $T < T_c$ and explain why this is in accordance with the figure from (ii).

(iv) Define the critical exponent β and determine the value predicted by the Landau theory for the Ising model.

4. Consider the diluted Ising model in zero external field with the energy

$$E_{\{s_i\}} = -\sum_{\langle i,j\rangle} J_{ij} s_i s_j,$$

where $s_i = \pm 1$ is the spin at lattice position i, the sum runs over different pairs of nearest neighbour sites, and the coupling constants

$$J_{ij} = \begin{cases} J > 0 & \text{with probability } p \\ 0 & \text{with probability } (1-p). \end{cases}$$

(i) Discuss how this problem is related to the theory of percolation.

(ii) In the following, assume the temperature $T = 0\text{K}$.

(a) What is the ground state of the diluted Ising model?

(b) Show that

$$\langle s_i s_j \rangle = \begin{cases} 1 & i \text{ and } j \text{ belong to the same percolation cluster} \\ 0 & \text{otherwise.} \end{cases}$$

(c) Argue why the average magnetisation per spin $m_0(p) = 0$ for $p \le p_c$. Based on your knowledge of percolation theory, find an expression for $m_0(p)$ when $p > p_c$.

(iii) You may assume, that for small non-zero external field H and low temperatures $k_B T \ll J$, the magnetisation per spin

$$m(p, H) = \pm P_\infty(p) + \sum_s s n(s, p) \tanh(sH/k_B T),$$

where $n(s, p)$ is the cluster number density

(a) What does $P_\infty(p)$ represent? Find the magnetisation $m_0(p)$ in the limit of $H \to 0$.

(b) Define the susceptibility χ_T. Assuming $H \ll k_B T$, show that the susceptibility diverges when $p \to p_c$.

(iv) A version of the fluctuation-dissipation theorem states that

$$\chi_T = \frac{1}{k_B T} \sum_i \sum_j \{\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle\}$$

where $\langle s_i \rangle = \langle s_j \rangle = m_0$. Assume p approaches p_c from below and $k_B T \ll J$. Calculate the susceptibility using this formula and show it is consistent with the result derived in (iii)(b).

Note: You may use the following expansion without proof $\tanh(x) = x - \mathcal{O}(x^3)$ *for* $x \ll 1$.

SECTION C

5. (i) Define explicitly the one-dimensional Bak-Tang-Wiesenfeld sandpile model in a lattice of size L.

(ii) (a) Describe the attractor of the dynamics.

(b) Define the avalanche size s and relate it to the energy dissipated during an avalanche.

(c) Assuming sand is deposited at random positions $i \in [1, L]$, show that the avalanche size probability is

$$P(s, L) = \begin{cases} 1/L & \text{for } 1 \leq s \leq L \\ 0 & \text{otherwise.} \end{cases}$$

(iii) Rewrite the avalanche size probability in the scaling form

$$P(s, L) = s^{-\tau} f(s/L^D).$$

Identify the critical exponents τ and D and the scaling function f.

(iv) (a) Sketch $P(s, L)$ versus s for three different system sizes $L = 25, 50$, and 100.

(b) Outline the procedure for making the data collapse onto one universal function and identify this function.

(v) Explain the concept of self-organised criticality using the one-dimensional Bak-Tang-Wiesenfeld sandpile model as an example. Why might the occurrence of earthquakes be related to the concept of self-organised criticality?

6. Not relevant for Statistical Mechanics 2006/2007

(i) Sketch the absolute value $|\Psi|$ of the order parameter for a superconducting phase transition as a function of temperature T. What is the physical interpretation of $|\Psi|$?

(ii) Consider a specimen displaying superconductivity.

(a) Explain the physical origin of the formation of Cooper pairs at low enough temperature T.

(b) Argue whether the formation of Cooper pairs increases or decreases the entropy S of the specimen?

(c) Argue whether the formation of Cooper pairs increase or decrease the internal energy U of the specimen?

(d) The thermodynamic definition of the free energy F is

$$F = \langle E \rangle - TS.$$

In view of your answers to questions (b) and (c) above argue why a phase transition will occur at low enough temperature T. Sketch the free energy F as a function of temperature T for a specimen in the superconducting state and in the normal state.

(iii) The phenomenological London equation states

$$\mathbf{j} = -\frac{1}{\mu_0 \lambda_L^2} \mathbf{A},$$

where $\mathbf{j}$ is the superconducting current density, $\mathbf{A}$ the vector potential (i.e., $\mathbf{B} = \nabla \times \mathbf{A}$), μ_0 the permeability, and λ_L the London penetration depth. Show that the superconducting current density is confined within a thickness λ_L of the surface of a superconductor.

(iv) Consider a specimen displaying type I superconductivity.

(a) Define the concept of critical external magnetic field $H_c(T)$.

(b) Assuming that $T < T_c$, sketch the magnetisation M versus the applied external magnetic field H.

(c) You are given a superconductor at $T < T_c$ in zero external magnetic field. What happens if the external magnetic field is increased to a value $H < H_c(T)$? Explain the physics of this phenomenon.

Note: You may use the following differential vector identity:
$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$.

CONDENSED MATTER THEORY EXAM MAY 2003

The paper consists of *three* sections: A & B & C.
Section A contains *two* questions.
Section B contains *two* questions.
Section C contains *two* questions.
All questions carry equal marks.

Answer THREE questions, taking ONE question from section A, ONE question from section B, and ONE question from section C.

SECTION A

1. Consider percolation in a lattice of infinite size $L = \infty$ with critical occupation probability p_c.

(i) (a) Define the order parameter $P_\infty(p, L = \infty)$ for the geometrical phase transition.

(b) Describe the behaviour of the order parameter as a function of p. Illustrate your explanation with a sketch.

(c) Let $n(s, p, L = \infty)$ denote the number of s-clusters per lattice site. Justify the relation

$$P_\infty(p, L = \infty) = p - \sum_{s=1}^{\infty} s n(s, p, L = \infty). \tag{1.1}$$

In the following $p > p_c$. Let $\xi(p)$ denote the correlation length and assume that the order parameter becoming non-zero for p approaching p_c from above is characterised by the critical exponent β, that is,

$$P_\infty(p, L = \infty) \propto (p - p_c)^\beta \quad \text{for } p \to p_c^+. \tag{1.2}$$

(i) (a) Show that for $p \to p_c^+$, $P_\infty(\xi, L = \infty) \propto \xi^{-\beta/\nu}$, where ν is the critical exponent characterising the divergence of the correlation length as $p \to p_c$.

(b) Argue why, for finite lattices $L < \infty$, with $L \ll \xi$ one would expect

$$P_\infty(\xi, L) \propto L^{-\beta/\nu}. \tag{1.3}$$

(c) Numerically, how would you determine the ratio $-\beta/\nu$?

(ii) In finite lattices, $L < \infty$, you may assume the cluster number density at $p = p_c$ obeys the scaling law (for all s)

$$n(s, p_c, L) = s^{-\tau} g(s/L^D), \tag{1.4}$$

where τ is the cluster number exponent, D is the fractal dimension and the scaling function $g(x)$ is constant for $x \ll 1$ and decays rapidly for $x \gg 1$. In addition, you may assume that

$$P_\infty(p_c, L) = \sum_{s=1}^{\infty} s\, s^{-\tau} g(0) - \sum_{s=1}^{\infty} s n(s, p_c, L). \tag{1.5}$$

(a) Show that Equation (1.5) is correct in the limit of $L \to \infty$.

(b) Combining Equations (1.3), (1.4) and (1.5) or otherwise, derive the scaling relation

$$-\beta/\nu = D(2 - \tau). \tag{1.6}$$

2. Consider site percolation on the square lattice in $d = 2$.

(i) (a) How many different microstates are associated with a 2×2 lattice?

(b) Sketch all the microstates and list the probability of each.

(ii) (a) Using cells of size $b \times b$ and adopting the rule of having a *vertically* spanning cluster to define a real space renormalisation transformation $R_b(p)$, show that

$$R_b(p) = 2p^2 - p^4, \tag{2.1}$$

when $b = 2$ and p denotes the occupation probability.

(b) Sketch the graph of the real space renormalisation transformation in Equation (2.1) and identify clearly the fixed points $p^\star$.

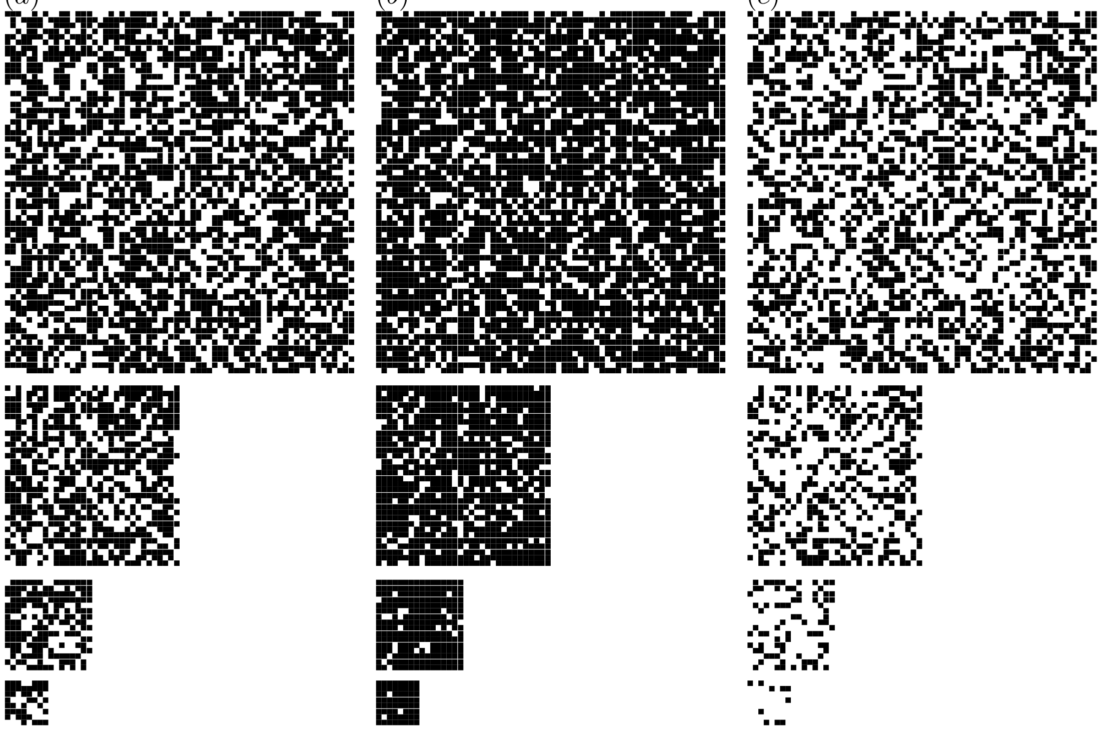
(c) Identify the critical occupation probability p_c and determine the correlation length exponent ν predicted by the real space renormalisation transformation.

In the diagrams below, lattices of size $L \times L = 64 \times 64$ with different initial occupation probabilities p_a, p_b, and p_c have been renormalised ($L \to L/2 \to L/4 \to L/8$) using the real space renormalisation transformation Equation (2.1).

(iii) (a) Discuss the sequences of renormalised lattices in diagrams (a), (b) and (c) in terms of the flow in p-space.

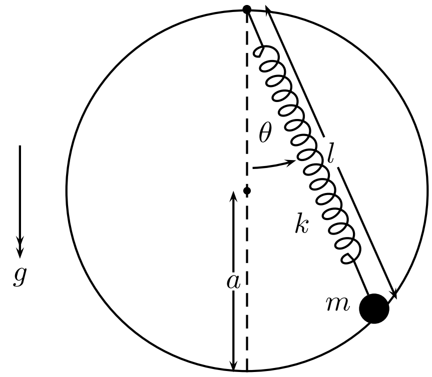
(b) Relate the concept of fixed points for a renormalisation group transformation to scale invariance.

(a) (b) (c)



SECTION B

3. A bead of mass m is suspended from a spring with spring constant k such that it rests inside a circular ring of radius a, which lies in the vertical plane. The acceleration due to gravity is g and you may assume there is no friction. Under the weight of the bead the spring extends from its natural length a to a length l. The spring makes an angle θ to be measured positive counterclockwise from the vertical. Geometry reveals that $l = 2a\cos\theta$.



(i) Show that the total energy of the mass-spring system is

$$U(\theta) = \frac{1}{2}ka^2(2\cos\theta - 1)^2 - mga\cos 2\theta,$$

where the zero of the gravitational potential energy is defined at the horizontal passing through the centre of the circle.

(ii) (a) Show that the system is in equilibrium when $\theta = 0$ or $\theta = \pm\arccos(\frac{ka}{2(ka-mg)})$.

(b) Show that $ka \geq 2mg$ for the existence of a non-trivial equilibrium angle θ.

Expanding the function $U(\theta)$ around $\theta = 0$ up to fourth order, we find that

$$U(\theta) = (\frac{1}{2}ka^2 - mga) + a(2mg - ka)\theta^2 + \frac{a}{12}(7ka - 8mg)\theta^4.$$

(iii) (a) Explain why only terms of even order appear in the expansion.

(b) Sketch the function $U(\theta) - (\frac{1}{2}ka^2 - mga)$ for $2mg > Ks$ and $2mg < ka$ assuming the coefficient of θ^4 is positive.

(c) Sketch the physical solution θ as a function of the ratio $\frac{2mg}{ka}$. Relate the graph to the sketch from (iii)(b).

(d) Briefly outline the Landau theory of second-order phase transitions for the ferromagnetic spin 1/2 Ising model.

(e) What is the order parameter of the mass-spring system? Explain your answer.

4. Not relevant for Statistical Mechanics 2006/2007

(i) The order parameter for a superconducting phase transition is a complex number

$$\Psi(\mathbf{r}) = |\Psi(\mathbf{r})| \exp(i\Theta(\mathbf{r})). \tag{4.1}$$

(a) What is the physical interpretation of $|\Psi(\mathbf{r})|^2$?

(b) Explain why one can assume that $|\Psi(\mathbf{r})|^2$ does not change significantly in space, that is, we can write $|\Psi(\mathbf{r})|^2 = |\Psi|^2$.

(ii) You may assume without proof the equation of continuity for probabilities

$$\mathbf{j}_{prob} = \frac{1}{2m}[\Psi^\star(\mathbf{r})(-i\hbar\nabla - q_s\mathbf{A})\Psi(\mathbf{r}) + \Psi(\mathbf{r})(i\hbar\nabla - q_s\mathbf{A})\Psi^\star(\mathbf{r})]. \tag{4.2}$$

(a) Identify clearly all the terms in the equation of continuity and explain why the right hand side is a real number.

(b) Derive the equation for the electrical current density

$$\mathbf{j} = \frac{q_s}{m}[\hbar\nabla\Theta(\mathbf{r}) - q_s\mathbf{A}]|\Psi|^2. \tag{4.3}$$

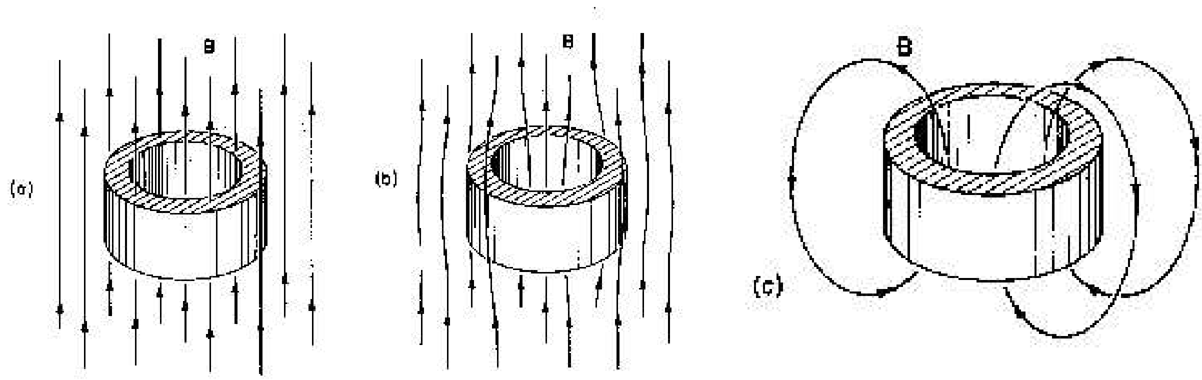
(c) What is the classical equivalent of Equation (4.3)?

Consider a superconducting ring.

(iii) (a) Describe the phenomenon of flux quantisation.

(b) Show that Equation (4.3) for the electrical current density implies flux quantisation.

(iv) Consider the diagrams below which displays a superconducting ring in an external magnetic field at temperature (a) $T > T_c$ and (b) $T < T_c$. In (c), the external magnetic field has been switched off. Explain what happens when going from (a) to (b) and from (b) to (c). Indicate whether you will observe flux quantisation in the settings (a), (b) and (c). What is the source of the magnetic loops in diagram (c)?



SECTION C

5. (i) (a) Write down the energy $E_{\{s_i\}}$ for a ferromagnetic spin 1/2 Ising model with coupling constant J and external field of strength H. Clearly identify all the symbols.

(b) Discuss the simplifications entering into the Ising model.

(c) Argue why such a simple model might be relevant for a real physical ferromagnet near the critical temperature T_c.

In the following, assume zero external magnetic field $H = 0$.

(ii) (a) Define the order parameter for the Ising model.

(b) Discuss the microscopic states and the associated value of the order parameter in the limits $T \to 0$ and $T \to \infty$.

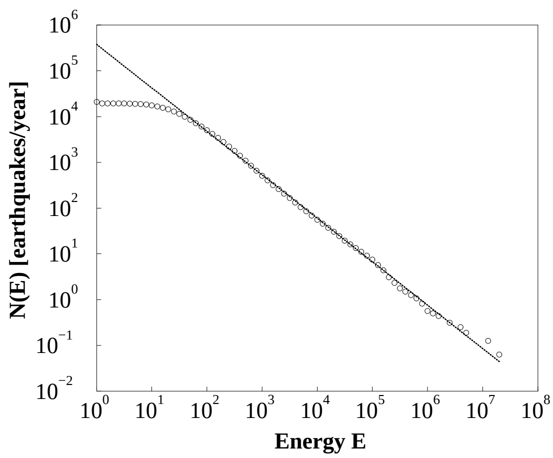
(c) Describe the behaviour of the order parameter as a function of T. Illustrate your explanation with a sketch. Relate the sketch to your answer in (ii)(b). Discuss qualitatively the microscopic states associated with $T = T_c$.

(iii) Given that the free free energy at temperature T is

$$F = \langle E \rangle - TS,$$

where where $\langle E \rangle$ is the internal energy and S the entropy, explain why you would expect the Ising model to exhibit a phase transition.

6. (i) The figure below displays the number of earthquakes $N(E)$ with energy release larger than E per year. Explain why this indicates that the seismic system might be viewed as being self-organised critical.



(ii) In the Ising model at the critical temperature $T = T_c$, the susceptibility $\chi = \left(\frac{\partial\langle M\rangle}{\partial H}\right)_T$ diverges. Contrast this system with the metaphor of a slowly driven sandpile, explaining briefly which sandpile quantity is the analogue of the susceptibility, and the difference between criticality in equilibrium systems and non-equilibrium systems displaying scale invariance.

In a model system displaying self-organised criticality, the avalanche size probability obeys (for all avalanche sizes s)

$$P(s, L) = s^{-\tau} f(s/L^D),$$

where τ and D are critical exponents, L is the system size and the scaling function $f(x)$ is constant (different from zero) for $x \ll 1$ and decays rapidly for $x \gg 1$.

(iii) (a) Assume $L = \infty$. Given that $P(s, L = \infty)$ can be normalised but that the average response $\langle s\rangle = \sum_{s=1}^{\infty} sP(s, \infty)$ diverges, derive lower and upper bounds for the critical exponent τ.

(b) Assuming L is finite, derive the scaling of the 4^{th} moment $\langle s^4\rangle = \sum_{s=1}^{\infty} s^4 P(s, L)$ with system size L in terms of τ and D.

(c) Numerically, how would you determine the graph of the scaling function f for the model system?

(iv) Why does it seem implausible that the scale invariance observed in equilibrium systems at a phase transition is related to the scale invariance observed in nature.

STATISTICAL PHYSICS EXAM MAY 2004

The paper consists of *three* sections: A & B & C.
Section A contains *two* questions.
Section B contains *two* questions.
Section C contains *two* questions.
All questions carry equal marks.

Answer THREE questions, taking ONE question from section A, ONE question from section B, and ONE question from section C.

SECTION A

1. Consider **one-dimensional** site percolation on an infinite lattice.

(i) (a) What is the critical occupation probability p_c? [1 marks]

(b) Determine the cluster number density $n(s,p)$, that is, the number of s clusters per lattice site at occupation probability p. [2 marks]

(c) Explain why

$$\sum_{s=1}^{\infty} sn(s,p) = p \quad \text{for } p < 1.$$

Why is this identity not valid for $p = 1$? [2 marks]

(d) Derive the identity

$$\sum_{s=1}^{\infty} s^2 p^s = p\frac{1+p}{(1-p)^3} \quad \text{for } p < 1.$$

[1 marks]

(e) Calculate the average cluster size

$$\chi(p) = \frac{\sum_{s=1}^{\infty} s^2 n(s,p)}{\sum_{s=1}^{\infty} sn(s,p)}.$$

Determine the amplitude Γ and the critical exponent γ such that

$$\chi(p) = \Gamma(p_c - p)^{-\gamma} \quad \text{for } p \to p_c^- .$$

[4 marks]

Consider one-dimensional site-bond percolation on an infinite lattice. Sites are occupied with probability p while bonds are occupied with probability q, see Figure 2. A cluster of size s is defined as having s consecutive occupied sites with $s-1$ intermediate occupied bonds. For example, the left-most cluster in Figure 2 has size $s = 3$. The cluster terminates to the right because a bond is empty and to the left because a site is empty.

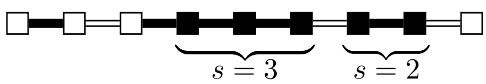


Figure 2: Part of an infinite one-dimensional lattice where occupied sites and bonds are black while empty sites and bonds are white. Two clusters are present with $s = 3$ and $s = 2$, respectively.

(ii) (a) What is the critical point, (p_c, q_c), for site-bond percolation? [2 marks]

(b) Show that the cluster number density

$$n(s,p,q) = p^s q^{s-1} (1-pq)^2 .$$

[4 marks]

(c) Calculate the average cluster size

$$\chi(p,q) = \frac{\sum_{s=1}^{\infty} s^2 n(s,p,q)}{\sum_{s=1}^{\infty} sn(s,p,q)}.$$

Comment on the result. [4 marks]

2. Consider percolation on a d-dimensional hypercubic lattice with critical occupation probability p_c. Let ξ denote the correlation length and let $M_\infty(\xi, \ell)$ denote the mass of the percolation cluster within a hypercubic box of linear size ℓ at occupation probability $p \geq p_c$.

(i) Sketch the behaviour of the correlation length for $p \geq p_c$ and discuss its geometrical interpretation. [2 marks]

(ii) (a) Explain why

$$M_\infty(\xi, \ell) \propto \ell^D \quad \text{for } \ell \ll \xi \tag{2.1}$$

and comment on the value of the exponent D. [2 marks]

(b) Show that

$$M_\infty(\xi, \ell) \propto \xi^D \, (\ell/\xi)^d \quad \text{for } \ell \gg \xi. \tag{2.2}$$

[4 marks]

(c) Combine Equations (2.1) and (2.2) into a scaling form for $M_\infty(\xi, \ell)$ and discuss the behaviour of the crossover function. [4 marks]

(d) Give a couple of examples from nature where there is a crossover from a fractal behaviour at small scales to a homogeneous behaviour at large scales. [2 marks]

(iii) (a) How is $M_\infty(\ell, \xi)$ related to the order parameter in percolation? [2 marks]

(b) Derive the hyper-scaling relation

$$D = d - \beta/\nu,$$

where the critical exponents β and ν characterise the pick up of the order parameter and the divergence of the correlation length at $p = p_c$, respectively. [4 marks]

SECTION B

3. Consider the two-dimensional Ising model on a square lattice in an external field H.

(i) (a) Outline the Kadanoff block-spin real-space renormalisation transition applied to the Ising model. [3 marks]

(b) Let $E_{\{s_i\}}$ denote the energy of the Ising model with N spins at reduced temperature t and reduced external field h and let $E'_{\{s_I\}}, N', t'$ and h' denote the corresponding variables in the renormalised lattice. Explain how to define implicitly the renormalised energy, $E'_{\{s_I\}}$, such that the partition function remains invariant under the real-space renormalisation transformation, that is,

$$Z(N, t, h) = Z(N', t', h'). \tag{3.1}$$

[2 marks]

Figure 3 consists of panels taken from infinite lattices of the Ising model in zero external field. In each of the three columns, the Kadanoff block-spin real-space renormalisation procedure has been carried out on systems with different initial temperatures T_1, T_2 and T_3.

(ii) (a) Discuss the sequences of renormalised lattices in columns (1), (2), and (3) in Figure 3 and the associated flow in the reduced nearest-neighbour coupling constant $K_1 = J/k_B T$. [3 marks]

(b) Describe qualitatively the microstates associated with the fixed points of the real-space renormalisation transformation. [3 marks]

(c) Relate the fixed points of the real-space renormalisation transformation to the correlation length and the concept of scale invariance. [2 marks]

Assuming that

$$t' = b^{y_t} t \quad \text{and} \quad h' = b^{y_h} h \quad \text{for } t \to 0^{\pm}, h \to 0, b > 0,$$

where b sets the scale of the renormalisation transformation and y_t, y_h are positive exponents, Equation (3.1) implies that the singular part of free energy per spin is a generalised homogeneous function,

$$f(t, h) = b^{-2} f(b^{y_t} t, b^{y_h} h) \quad \text{for } t \to 0^{\pm}, h \to 0, \text{ for all } b > 0. \tag{3.2}$$

(iii) (a) Explain how the exponent y_t is related to the critical exponent ν characterising the divergence of the correlation length at the critical point $(t, h) = (0, 0)$. [3 marks]

(b) By choosing an appropriate factor b, show that Equation (3.2) implies the Widom scaling ansatz for the free energy

$$f(t, h) = |t|^{2-\alpha} \mathcal{F}_{\pm}(h/|t|^{\Delta}) \quad \text{for } t \to 0^{\pm}, h \to 0$$

and identify the gap exponent Δ and the scaling functions $\mathcal{F}_{\pm}$. [4 marks]

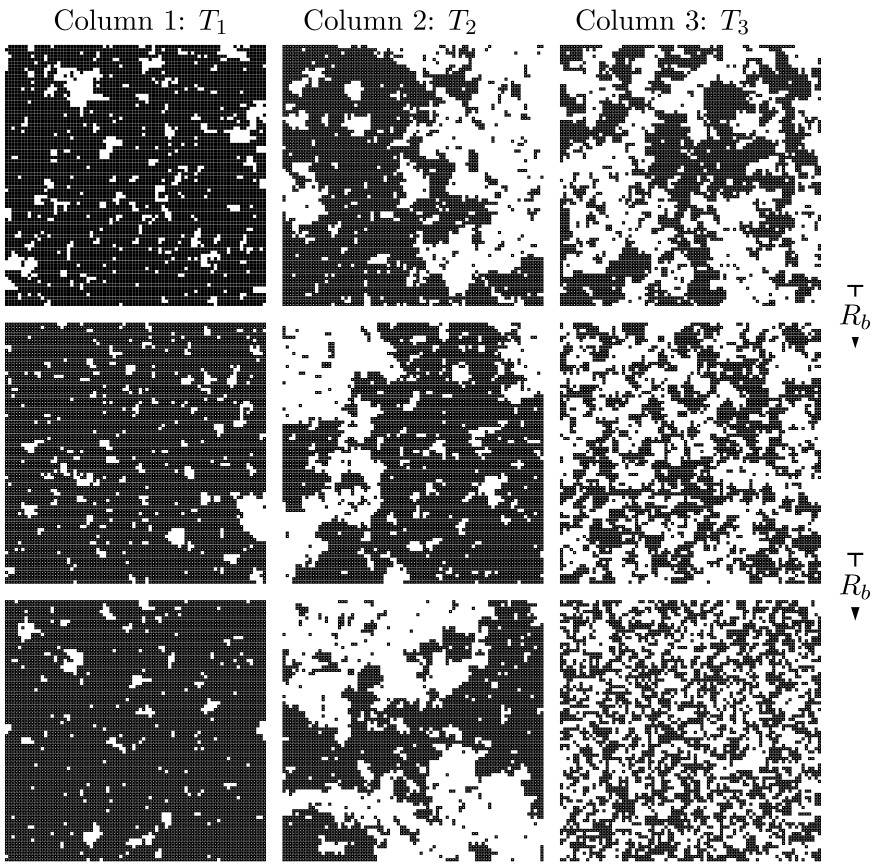


Figure 3: Real-space renormalisation of the Ising model on a two-dimensional square lattice in zero external field. In each of the three columns, the renormalisation transformation R_b is carried out twice from top to bottom.

4. Consider the Ising model in an external field H.

(i) (a) Write down the energy $E_{\{s_i\}}$ for the Ising model. Clearly identify all symbols. [2 marks]

(b) Define the order parameter for the Ising model. [2 marks]

(c) By considering the relative importance of the energetic and entropic contributions of the free energy

$$F = \langle E \rangle - TS,$$

discuss qualitatively the behaviour of the order parameter as a function of temperature T in zero external field. [4 marks]

(ii) (a) Define the susceptibility per spin, χ, and show that the susceptibility is related to the variance of the total magnetisation

$$k_B T \chi = \frac{1}{N} \left(\langle M^2 \rangle - \langle M \rangle^2 \right).$$

[3 marks]

(b) Let $\mathbf{r}_i$ denote the position of spin s_i. Define the site-site correlation function $g(\mathbf{r}_i, \mathbf{r}_j)$ and hence show that

$$k_B T \chi = \sum_{j=1}^{N} g(\mathbf{r}_i, \mathbf{r}_j). \qquad (4.1)$$

[3 marks]

Let t and h denote the reduced temperature and external field, respectively. You may assume the following scaling form of the spin-spin correlation function in the vicinity of the critical point

$$g(r, t, h) \propto r^{-(d-2+\eta)} \mathcal{G}_{\pm}(r/\xi, h/|t|^{\Delta}) \quad \text{for } t \to 0^{\pm}, h \to 0, \qquad (4.2)$$

where $r = |\mathbf{r}_i - \mathbf{r}_j|$ is the distance between the spins, d is the dimension, $\eta \geq 0$ is a critical exponent, Δ is the gap exponent, ξ is the correlation length and $\mathcal{G}_+$ and $\mathcal{G}_-$ are the scaling functions in the two regimes $t > 0$ and $t < 0$, respectively.

(iii) (a) Replacing the sum on the right-hand side of Equation (4.1) with an integral, explain why the scaling form of the site-site correlation function in Equation (4.2) implies the divergence of the susceptibility per spin in zero external field when $t \to 0$. [2 marks]

(b) Derive a scaling relation between η and the critical exponents γ and ν describing, respectively, the divergence of the susceptibility per spin and the correlation length at $(t, h) = (0, 0)$. [4 marks]

SECTION C

5. Consider a Bak-Tang-Wiesenfeld type sandpile model defined on a tree-like structure with N sites, $i = 1, \ldots, N$, each having h_c downward neighbours, see Figure 4. Each site i is assigned an integer h_i. Grains are added at random sites $h_i \rightarrow h_i + 1$. When $h_i \geq h_c$, the site i topples and one grain is added to each of the h_c downward neighbours.

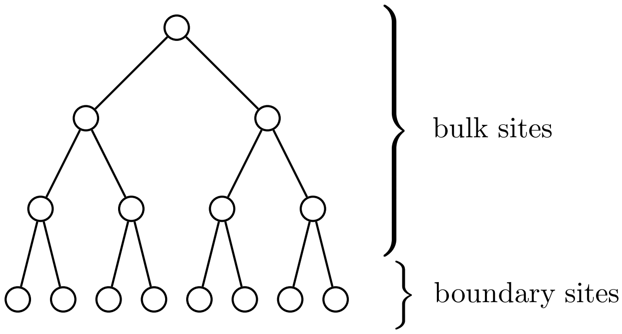


Figure 4: A tree-like structure with $h_c = 2$. The bottom most sites are boundary sites. The remaining sites are bulk sites.

The algorithm for the dynamics is defined as:

1. Place the pile in an arbitrary stable configuration with $h_i < h_c$ for all sites i.

2. Add a grain at a random site i, that is, $h_i \rightarrow h_i + 1$.

3. If $h_i \geq h_c$, the site relaxes and

$$h_i \rightarrow h_i - h_c$$
$$h_j \rightarrow h_j + 1 \quad \text{for the } h_c \text{ downward neighbours}$$

except when boundary sites topple, where,

$$h_i \rightarrow h_i - h_c.$$

A stable configuration is reached when $h_i < h_c$ for all sites i.

4. Proceed to step **2.** and reiterate.

(i) (a) How many stable configurations are there in total? [2 marks]

(b) Discuss briefly the concept of transient and recurrent configurations in sandpile models in general. [2 marks]

(c) Explain why both the minimally stable configuration with $h_i = h_c - 1$ for all sites i and the empty configuration with $h_i = 0$ for all sites i are recurrent states. Hence argue why all stable configurations are recurrent configurations on this tree-like structure. [2 marks]

(ii) (a) Let P_h denote the probability that a site contains h grains. The probability that the addition of a single grain to a stable configuration at a random site in the bulk will cause b of its h_c downward neighbours to topple is

$$p_b = \binom{h_c}{b} P_{h_c-1}^b \left(1 - P_{h_c-1}\right)^{h_c-b} \quad b = 0, \dots, h_c.$$

Justify this result. [2 marks]

(b) Argue why the average number of new topplings is

$$\langle b \rangle = h_c P_{h_c-1}.$$

[2 marks]

(iii) You may assume that in a typical stable configuration

$$P_h = P_{h-1} \quad \text{for } h = 1, \dots, h_c - 1.$$

Hence show that the average number of new topplings

$$\langle b \rangle = 1$$

and comment on the result. [3 marks]

(iv) (a) Consider a tree with $h_c = 2$, see Figure 4. Discuss how this model is related to percolation on a Bethe lattice with coordination number $z = 3$. [4 marks]

(b) Using an argument similar to that for deriving the average cluster size in percolation on a Bethe lattice, show that the average avalanche size in an infinite tree with $h_c = 2$ is

$$\langle s \rangle = \frac{P_{h_c-1}}{1 - h_c P_{h_c-1}}.$$

[3 marks]

6. Consider a one-dimensional granular pile on a finite horizontal base of length L confined between two glass plates. There is a vertical wall at the left boundary. Grains can leave the system at the right boundary.

(i) Using the metaphor of a one-dimensional granular pile driven by adding grains at the left boundary, explain briefly the concept of self-organised criticality. [4 marks]

(ii) (a) Define explicitly the dynamical algorithm for the one-dimensional Oslo rice pile model driven at the left boundary. [3 marks]

(b) For the Oslo rice pile model, explain the concept of recurrent configurations. [2 marks]

The one-dimensional Oslo model of linear size L displays self-organised criticality and the probability density of the avalanche size s satisfies

$$P(s, L) \propto s^{-\tau} \mathcal{G}(s/L^D) \quad \text{for } s \gg 1, L \gg 1, \tag{6.1}$$

where D is the avalanche dimension and τ the avalanche size exponent. The scaling function $\mathcal{G}(x)$ is a non-zero constant for $x \ll 1$ and decays rapidly for $x \gg 1$.

(iii) Assuming that Equation (5.1) is valid for all s, derive the scaling of the kth moment,

$$\langle s^k \rangle = \sum_{s=1}^{\infty} s^k P(s, L),$$

with system size $L \gg 1$ in terms of the exponents D and τ. [4 marks]

Numerical measurement of the kth moment in a variant of the Oslo model yields

Moment	$L = 100$	$L = 400$	$L = 1600$
$\langle s \rangle$	3.38×10^3	5.35×10^4	8.54×10^5
$\langle s^2 \rangle$	3.65×10^7	1.21×10^{10}	4.16×10^{12}
$\langle s^4 \rangle$	8.76×10^{15}	1.35×10^{21}	2.28×10^{26}
$\langle s^6 \rangle$	3.61×10^{24}	2.73×10^{32}	2.31×10^{40}

(iv) (a) Show that the numerical measurements of the first moment in this variant of the Oslo model are consistent with

$$\langle s \rangle \propto L^2 \quad \text{for } L \gg 1,$$

and hence derive a scaling relation between D and τ. [4 marks]

(b) Using the numerical measurements of your choice, determine the critical exponents D and τ. [3 marks]

STATISTICAL PHYSICS EXAM MAY 2005

The paper consists of *three* sections: A & B & C.
Section A contains *two* questions.
Section B contains *two* questions.
Section C contains *two* questions.
All questions carry equal marks.

Answer THREE questions, taking ONE question from section A, ONE question from section B, and ONE question from section C.

SECTION A

1. (i) Consider *one-dimensional* site percolation on an *infinite* lattice.

(a) What is the critical occupation probability p_c? Explain your answer.

(b) Define the correlation length, $\xi(p)$, at occupation probability p and show that

$$\xi(p) = -\frac{1}{\ln p}.$$

(c) Hence show that

$$\xi(p) \to (p_c - p)^{-\nu} \quad \text{for } p \to p_c^-,$$

and identify the critical exponent ν.

(d) Let $\Pi_\infty(p, L = \infty)$ denote the probability of having a percolating cluster at occupation probability p in an infinite lattice. Describe the behaviour of the function $\Pi_\infty(p, L = \infty)$ as a function of p. Illustrate your explanation with a sketch of the graph.

(ii) Consider *one-dimensional* site percolation on a *finite* lattice.

(a) Let $\Pi_\infty(p; L)$ denote the probability of having a percolating cluster in a lattice of size L. What is the function $\Pi_\infty(p; L)$? Explain your answer and sketch the graph of the function.

(b) If ξ is the correlation length at occupation probability p, show that

$$\Pi_\infty(\xi; L) = \exp(-L/\xi).$$

(c) Hence show that

$$\Pi_\infty(p; L) = \mathcal{F}_{1\text{d}}\left[(p_c - p)L\right] \quad \text{for } p \to p_c^-,$$

and identify the scaling function $\mathcal{F}_{1\text{d}}$. How does $\mathcal{F}_{1\text{d}}(x)$ behave for $x \ll 1$ and $x \gg 1$?

(iii) In higher dimensions there exists a scaling function $\mathcal{G}$ such that

$$\Pi_\infty(\xi; L) = \mathcal{G}(L/\xi) \quad \text{for } p \to p_c.$$

(a) Hence show that

$$\Pi_\infty(p; L) = \mathcal{F}\left[|p_c - p|L^{1/\nu}\right] \quad \text{for } p \to p_c,$$

and relate the scaling function $\mathcal{F}$ to $\mathcal{G}$.

(b) What is the limiting function $d\Pi_\infty/dp$ when $L \to \infty$?

2. In bond percolation, each bond between neighbouring lattice sites is occupied with probability p and empty with probability $(1-p)$. The bond percolation threshold for a square lattice $p_c = 0.5$. In a real-space renormalisation procedure on the square lattice with unit lattice spacing, the lattice is replaced by a new renormalised lattice, with super-bonds of length $b = 2$ occupied with probability $R_b(p)$, following the procedure shown in the figure below.

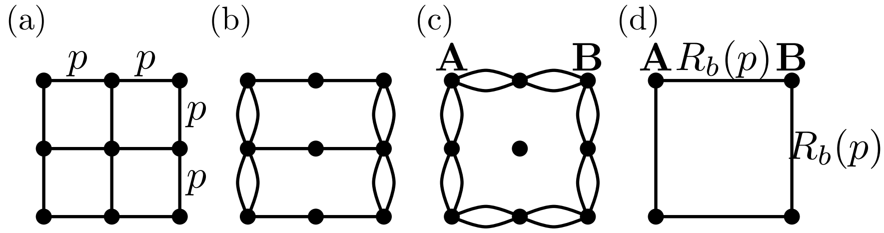


Figure 5: (a) Original lattice with unit lattice spacing where each bond is occupied with probability p. (b) Lattice where every second column in the original lattice is moved one lattice unit to the left. (c) Lattice where, in addition, every second row in the original lattice is moved one lattice unit upwards. In this lattice, there are two bonds between each site. (d) Renormalised lattice with lattice spacing $b = 2$ where each super-bond is occupied with probability $R_b(p)$.

(i) Assuming that the super-bond between **A** and **B** in the renormalised lattice is occupied if there exists a connected path from **A** to **B** along the four bonds in lattice (c), show that

$$R_b(p) = p^4 - 4p^3 + 4p^2. \tag{2.1}$$

(ii) (a) Solve graphically the fixed point equation for the renormalisation group transformation in Equation (2.1).

(b) Describe the flow in p-space and the renormalisation of the correlation length when applying the renormalisation group transformation repeatedly.

(c) Identify clearly the correlation lengths associated with the fixed points $p^\star$ of the renormalisation group transformation and hence explain why fixed points are associated with scale invariance.

(iii) (a) Derive a form for the critical exponent ν in terms of the renormalisation group transformation.

(b) Hence, identify the critical occupation probability p_c and determine the correlation length exponent ν predicted by the renormalisation group transformation in Equation (2.1).

SECTION B

3. Consider the Ising model on a d-dimensional lattice in an external field H.

(i) (a) Write down the energy $E_{\{s_i\}}$ for the Ising model. Clearly identify all symbols.

(b) Identify the order parameter for the Ising model and discuss qualitatively its behaviour as a function of temperature T in zero external field.

(ii) Let $t = (T - T_c)/T_c$ and $h = H/k_B T$ denote the reduced temperature and external field, respectively. Assume that the singular part of free energy per spin is a generalised homogeneous function,

$$f(t, h) = b^{-d} f(b^{y_t} t, b^{y_h} h) \quad \text{for } t \to 0^{\pm}, h \to 0, b > 0, \tag{3.1}$$

where d is the dimension and y_t, y_h are positive exponents.

(a) Define the critical exponent α associated with the specific heat in zero external field and show that Equation (3.1) implies

$$\alpha = \frac{2y_t - d}{y_t}.$$

(b) Define the critical exponent β associated with the order parameter in zero external field and show that Equation (3.1) implies

$$\beta = \frac{d - y_h}{y_t}.$$

(c) Define the critical exponent γ associated with the susceptibility in zero external field and show that Equation (3.1) implies

$$\gamma = \frac{2y_h - d}{y_t}.$$

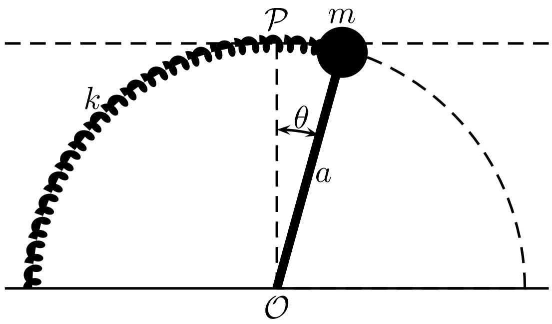
(d) Define the critical exponent δ associated with the order parameter at the critical temperature and show that Equation (3.1) implies

$$\delta = \frac{y_h}{d - y_h}.$$

(e) Hence confirm the two scaling relations

$$\alpha + 2\beta + \gamma = 2,$$
$$\gamma = \beta(\delta - 1).$$

4. A rigid massless rod of length a can rotate around a fixed point $\mathcal{O}$ in the vertical plane only. The orientation of the rod is given by its angle θ to be measured positive clockwise from the vertical. At the top of the rod is placed a *variable* mass m which is linked to a circular harmonic spring of radius a and spring constant k. When the rod is vertical, the length of the spring equals its natural length, $\pi a/2$.



(i) Show that the total energy of the mass-spring system is

$$U(\theta) = \frac{1}{2}ka^2\theta^2 + mga(\cos\theta - 1), \tag{4.1}$$

where the zeroth-level of the gravitational potential energy is defined as the horizontal dashed line passing through the point $\mathcal{P}$, the position of the mass when the rod is vertical.

(ii) (a) Expand the function $U(\theta)$ in Equation (4.1) around $\theta = 0$ up to fourth order, to show that

$$U(\theta) = \frac{a}{2}(ka - mg)\theta^2 + \frac{mga}{24}\theta^4. \tag{4.2}$$

(b) Explain why only terms of even order appear in the expansion in Equation (4.2).

(c) Sketch the function $U(\theta)$ in Equation (4.2) for $ka > mg, ka = mg$, and $ka < mg$.

(d) Using Equation (4.2), find an explicit expression for the angle of equilibrium $\theta_0(m)$ when $ka > mg$ and $ka < mg$.

(e) Sketch the solution of the angle of equilibrium $\theta_0(m)$ as a function of the ratio ka/mg. Relate the graph to the sketches from (c).

(f) Briefly outline the Landau theory of second-order phase transitions in general.

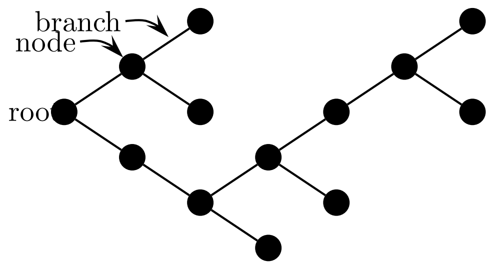
(g) What is the order parameter of the mass-spring system? What is the critical value, m_c, of the variable mass m? Explain your answers.

SECTION C

5. Consider a slowly driven 'sandpile like' system of size L which eventually reaches a set of recurrent configurations. Let a sequence of consecutive toppling sites be identified with an avalanche.

(i) Which quantity is a measure of the susceptibility of the system?

(ii) Explain how a finite avalanche might be visualised by a finite rooted tree.



(iii) (a) Define the average branching ratio, $\langle b \rangle$, for an ensemble of finite avalanches and and show that

$$\langle b \rangle = 1 - \frac{1}{\langle s \rangle}$$

where $\langle s \rangle$ is the average size of an avalanche.

(b) If the 'sandpile like' system displays self-organised criticality, what is the limiting value of $\langle b \rangle$ when $L \to \infty$?

(iv) (a) Define explicitly the one-dimensional Bak-Tang-Wiesenfeld sandpile model in a lattice of size L driven by adding grains at random positions $i \in [1, L]$.

(b) Discuss briefly the concept of transient and recurrent configurations and describe the set of recurrent configurations in the one-dimensional Bak-Tang-Wiesenfeld sandpile model.

(c) What is the average branching ratio, $\langle b \rangle$, for the one-dimensional Bak-Tang-Wiesenfeld sandpile model in a lattice of size L? Does the limiting value of $\langle b \rangle$ when $L \to \infty$ agree with your answer to question (iii)(b)?

6. Consider a system of size L which displays self-organised criticality. Let s denote the avalanche size with a corresponding avalanche-size probability

$$P(s; L) = as^{-\tau}\mathcal{G}\left(s/bL^{D}\right) \quad \text{for } s \gg 1, L \gg 1 \tag{6.1}$$

where τ and D are universal critical exponents and a and b are non-universal constants. The scaling function $\mathcal{G}(x)$ is a non-zero constant for $x \ll 1$ and decays rapidly for $x \gg 1$.

(i) Assume that the avalanche-size probabilities are measured in systems of sizes $L_4 \gg L_3 \gg L_2 \gg L_1 \gg 1$. Explain how to make a data collapse of the graphs for the avalanche-size probabilities. Illustrate your explanation with a sketch, clearly labelling your axes.

(ii) In the following you may assume that Equation (6.1) is valid for all avalanche sizes.

(a) Keeping track of the non-universal constants, calculate the scaling with system size L of the kth moment of the avalanche-size probability

$$\langle s^k \rangle = \sum_{s=1}^{\infty} s^k P(s; L) \equiv \Gamma_k L^{\gamma_k} \quad \text{for } L \gg 1,$$

and identify the critical exponent γ_k and the amplitude Γ_k. Comment whether they are universal.

(b) Show that the moment ratio

$$g_k = \frac{\langle s^k \rangle \langle s \rangle^{k-2}}{\langle s^2 \rangle^{k-1}}$$

is a universal quantity.

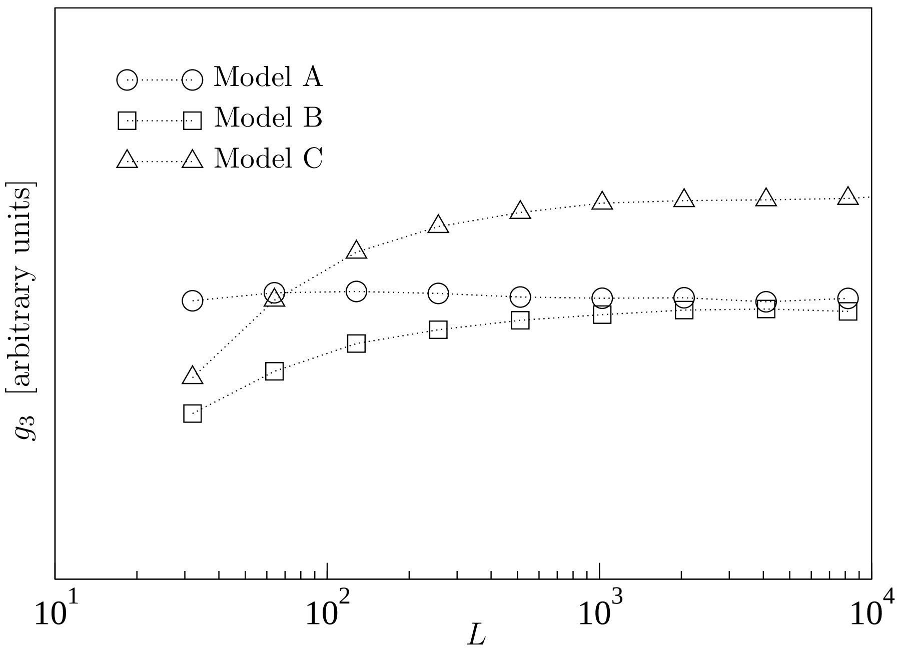


Figure 6: The numerical measurements of moment ratio g_3 as a function of system size L for three models A, B, and C.

(iii) Consider three models A, B, and C displaying self-organised criticality in which g_3 has been measured numerically as a function of system size.

(a) Taking the numerical measurements of g_3 for Model C as an example, argue qualitatively, why you would expect g_3 to be constant only for $L \to \infty$.

(b) Considering that g_3 is independent of non-universal constants, what can you conclude from the data about the three models with respect to universality classes?

STATISTICAL PHYSICS EXAM MAY 2006

The paper consists of *three* sections: A & B & C.
Section A contains *two* questions.
Section B contains *two* questions.
Section C contains *two* questions.
All questions carry equal marks.

Answer THREE questions, taking ONE question from section A, ONE question from section B, and ONE question from section C.

SECTION A

1. (i) Consider percolation on an *infinite* lattice with critical occupation probability p_c.

(a) Sketch the average cluster size, $\chi(p)$, versus occupation probability, p, and comment on the graph. [2 marks]

(b) Define the critical exponent γ associated with $\chi(p)$. [2 marks]

(c) Hence show that

$$\chi(p) \propto \xi^{\gamma/\nu} \quad \text{for } p \to p_c,$$

where $\xi(p)$ is the correlation length and ν its associated critical exponent. [2 marks]

(ii) Consider percolation on *finite* lattices.

(a) Assume $p = p_c$. Sketch the logarithm of the average cluster size, $\log \chi(p_c; L)$, versus the logarithm of the lattice size, $\log L$. Explain the form of the graph in your sketch. [4 marks]

(b) Assume $p \approx p_c$ with $p \neq p_c$. Sketch the logarithm of the average cluster size, $\log \chi(p; L)$, versus the logarithm of the lattice size, $\log L$. Explain the form of the graph in your sketch. [4 marks]

(c) You may assume the scaling ansatz

$$n(s, p_c) = s^{-\tau} \mathcal{G}\left(s/L^D\right) \quad \text{for } p \to p_c, L \gg 1 \tag{1.1}$$

for the cluster number density at $p = p_c$ is valid for all cluster sizes s. Calculate the average cluster size using the scaling ansatz Equation (1.1) and derive the scaling of the average cluster size with lattice size L. [4 marks]

(d) Hence derive the scaling relation

$$\gamma/\nu = D(3 - \tau).$$

[2 marks]

2. Consider site percolation on the square lattice in two dimensions and let p denote the occupation probability.

(i) Sketch all the microstates associated with a 2×2 block and list the associated probability of each microstate. [2 marks]

(ii) Outline the real-space renormalisation group procedure applied to site percolation in two dimensions when coarse-graining over scale $b = 2$. [4 marks]

(iii) (a) Using blocks of size $b \times b$ and adopting the rule of having a *diagonally* spanning cluster to define the renormalisation group transformation $R_b(p)$, show that

$$R_b(p) = 4p^3 - 3p^4, \tag{2.1}$$

when $b = 2$. [2 marks]

(b) Solve graphically or otherwise the fixed point equation for the renormalisation group transformation in Equation (2.1). [2 marks]

(c) Describe the flow in p-space and the renormalisation of the correlation length when applying the renormalisation group transformation repeatedly and identify clearly the correlation lengths associated with the fixed points $p^\star$. [4 marks]

(iv) (a) Derive a form for the critical exponent ν in terms of the renormalisation group transformation $R_b(p)$. [4 marks]

(b) Hence, identify the critical occupation probability p_c and determine the correlation length exponent ν predicted by the renormalisation group transformation in Equation (2.1). [2 marks]

SECTION B

3. Consider the Ising model in an external field H and let $m = m(T, H)$ denote the average magnetisation per spin at temperature T. The starting point for the Landau theory of the Ising model is the free energy per spin on the form

$$f(T, H; m) = f_0 - Hm + a_2(T - T_c)m^2 + a_4 m^4, \tag{3.1}$$

where f_0 is the entropic part of the free energy, T_c the critical temperature and a_2, a_4 positive constants.

(i) (a) Explain why the free energy per spin $f(T, H; m)$ can be expanded in the average magnetisation per spin in the vicinity of a critical point $(T, H) = (T_c, 0)$. [2 marks]

(b) Argue why the expansion for $f(T, H; m)$ only contains even powers of m in zero external field. [2 marks]

(ii) (a) Sketch the free energy per spin in zero external field as a function of magnetisation per spin $m_0(T) = m(T, 0)$ for temperatures $T > T_c$, $T = T_c$ and $T < T_c$. [4 marks]

(b) Sketch the magnetisation per spin in zero external field, $m_0(T)$, as a function of temperature, T, as predicted by the Landau theory of the Ising model. Explain the sketch is related to your graphs in (a). [4 marks]

(iii) (a) Define the order parameter for the Ising model and its associated critical exponent β. [2 marks]

(b) Derive the value of the critical exponent β predicted by the Landau theory of the Ising model. [4 marks]

(iv) Explain whether the critical exponents predicted by the Landau theory of the Ising model are universal or non-universal. [2 marks]

4. Consider an Ising model in dimension $d > 1$.

(i) (a) Write down the energy $E_{\{s_i\}}$ for the Ising model with N spins in an external field H. Clearly identify all symbols. [4 marks]

(b) Discuss the simplifications entering into the Ising model. [2 marks]

(ii) The partition function is given by $Z = \sum_{\{s_i\}} \exp\left(-\beta E_{\{s_i\}}\right)$ where $\beta = 1/(k_B T)$ is the inverse temperature.

(a) Express the free energy per spin $f(T, H)$ in terms of the partition function. [2 marks]

(b) Define the average magnetisation per spin $m(T, H)$ for the Ising model and show that

$$m(T, H) = -\left(\frac{\partial f}{\partial H}\right)_T .$$

[3 marks]

(iii) Make a sketch of the average magnetisation per spin in zero external field, $m_0(T) = \lim_{H \to 0^\pm} m(T, H)$ as a function of temperature, T. Explain your answer. [4 marks]

(iv) Figure 7 below displays the free energy per spin, $f(T, H)$, as a function of external field for, H, for three temperatures $T > T_c$, $T = T_c$ and $T < T_c$.

(a) How can you determine the average magnetisation per spin graphically? [2 marks]

(b) Consider the average magnetisation per spin in the limit $H \to 0^\pm$. Explain whether your sketch in subquestion (iii) is qualitatively consistent with the average magnetisation per spin $lim_{H \to 0^\pm} m(T, H)$ determined from Figure 7. [3 marks]

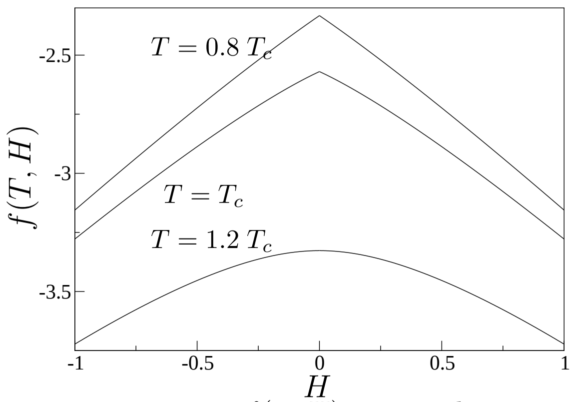


Figure 7: The free energy per spin, $f(T, H)$, versus the external field, H, for temperatures $T = 1.2\ T_c$ (lower graph) $T = T_c$ (middle graph) and $T = 0.8\ T_c$ (upper graph).

SECTION C

5. Consider a system of size $L \gg 1$ which displays self-organised criticality. Let s denote the avalanche size with a corresponding avalanche-size probability

$$P(s; L) = \begin{cases} as^{-\tau} & \text{for } 1 \leq s \leq L^D \\ 0 & \text{otherwise,} \end{cases} \tag{5.1}$$

where τ is the avalanche-size exponent and D the avalanche dimension.

(i) (a) Derive an inequality for τ such that the average avalanche size $\langle s \rangle$ diverges with system size. [2 marks]

(b) The critical exponent γ is defined by

$$\langle s \rangle \propto L^\gamma \quad \text{for } L \to \infty.$$

Determine a scaling relation between γ, τ and D. [2 marks]

(ii) In the following, assume that $\tau > 1$ for the avalanche-size probability given in Equation (5.1).

(a) Show that for $L \gg 1$, the parameter $a \approx (\tau - 1)$. [3 marks]

(b) Using the Heaviside step function

$$\Theta(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{otherwise,} \end{cases} \tag{5.2}$$

show that the avalanche-size probability in Equation (5.1) satisfies the scaling form

$$P(s; L) = as^{-\tau} \mathcal{G}\left(s/L^D\right) \quad \text{for } L \gg 1 \tag{5.3}$$

and identify the scaling function $\mathcal{G}$. [5 marks]

(iii) In the following, assume that $\tau < 1$ for the avalanche-size probability given in Equation (5.1).

(a) Show that for $L \gg 1$, the parameter $a \approx (1 - \tau)/L^{D(1-\tau)}$. [3 marks]

(b) Using the Heaviside step function in Equation (5.2) show that the avalanche-size probability in Equation (5.1) satisfies the scaling form

$$P(s; L) = \tilde{a}s^{-1} \tilde{\mathcal{G}}\left(s/L^D\right) \quad \text{for } L \gg 1 \tag{5.4}$$

and identify the scaling function $\tilde{\mathcal{G}}$ and $\tilde{a}$. [5 marks]

6. (i) Without going into details, explain briefly the hallmarks of self-organised criticality. [4 marks]

Consider a system of size $L \gg 1$ which displays self-organised criticality. Let s denote the avalanche size with a corresponding avalanche-size probability

$$P(s;L) = as^{-\tau}\mathcal{G}\left(s/bL^D\right) \quad \text{for } s \gg 1, L \gg 1 \tag{6.1}$$

where τ and D are critical exponents and a, b are constants. Figure 8 displays the measured avalanche-size probabilities for three systems of size $L_1, L_2, L_3 \gg 1$.

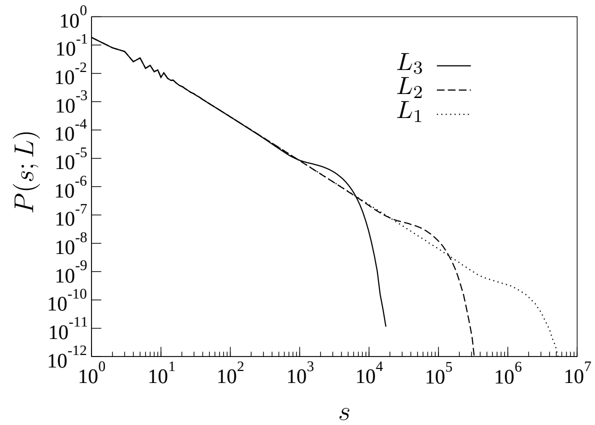


Figure 8: Numerical results for the avalanche-size probability, $P(s;L)$, versus the avalanche size, s, in three systems of size L_1, L_2 and L_3.

(ii) (a) Discuss the significance of the ansatz in Equation (6.1). [4 marks]

(b) Identify whether a, b, τ, D and $\mathcal{G}(x)$ in Equation (6.1) are universal or non-universal and briefly discuss the concept of universality. [3 marks]

(c) Discuss the concept of cut-off avalanche sizes with reference to Equation (6.1). Using the data in Figure 8, identify roughly the numerical value of the cut-off avalanche size in each of the three systems. [2 marks]

(d) Rank the system sizes in increasing order. Explain your answer. [2 marks]

(e) How would you determine the exponents τ and D if given the data in Figure 8. Accompany your answer with sketches if necessary. [3 marks]

(f) In some publications, the exponent τ is measured as the slope of the graph of $P(s;L)$ as displayed in Figure 8. Discuss why this might not always be correct e.g. by giving a counter-example. [2 marks]