**7.** *The order parameter on a Bethe lattice with coordination number z*

(i) $P_\infty(p)$ is the probability that an arbitrarily selected site belongs to the percolating infinite cluster. Consider the 'origin' in the Bethe lattice.

$$\begin{aligned} P_\infty(p) &= \text{probability 'origin' is occupied } \cdot \\ &\quad \text{probability } \textit{at least one} \text{ of the } z \text{ branches connects to infinity} \\ &= p\,[1 - Q_\infty^z(p)] \end{aligned} \tag{7.1}$$

where $Q_\infty(p)$ denotes the probability that a given branch does *not* connect to infinity. Again, we will rely on the fact that all sites in a Bethe lattice are equivalent, so $Q_\infty(p)$ is also the probability that a subbranch does not connect to infinity. Hence

$$\begin{aligned} Q_\infty(p) &= \text{neighbour to 'origin' is empty } + \text{ neighbour to 'origin' is occupied} \\ &\quad \text{but } \textit{none} \text{ of the } (z-1) \text{ subbranches connect to infinity} \\ &= (1-p) + pQ_\infty^{z-1}(p) \end{aligned} \tag{7.2}$$

(ii) For convenience, we drop the $p$-dependence of $Q_\infty(p)$ and simply write $Q_\infty$. Let

$$Q_\infty^{z-1} = (1 - [1 - Q_\infty])^{z-1} = (1-x)^{z-1} \quad \text{with } x = 1 - Q_\infty.$$

We expand to second order in $x$ around $x = 0$.

$$\begin{aligned} f(x) &= (1-x)^{z-1} && \Rightarrow f(0) = 1 \\ f^{(1)}(x) &= -(z-1)(1-x)^{z-2} && \Rightarrow f^{(1)}(0) = -(z-1) \\ f^{(2)}(x) &= (z-1)(z-2)(1-x)^{z-3} && \Rightarrow f^{(2)}(0) = (z-1)(z-2) \end{aligned} \tag{7.3}$$

implying

$$\begin{aligned} Q_\infty^{z-1} &\approx f(0) + f^{(1)}(0)x + \frac{1}{2}f^{(0)}(0)x^2 + \ldots \\ &= 1 - (z-1)x + \frac{1}{2}(z-1)(z-2)x^2 + \ldots \\ &= 1 - (z-1)(1-Q_\infty) + \frac{1}{2}(z-1)(z-2)(1-Q_\infty)^2 + \ldots. \end{aligned} \tag{7.4}$$

Using the Taylor expansion result in Equation (7.2) we find

$$\begin{aligned} Q_\infty &= 1 - p + pQ_\infty^{z-1} \\ &\approx 1 - p + p - p(z-1)(1-Q_\infty) + \overbrace{p\frac{1}{2}(z-1)(z-2)}^{a}(1-Q_\infty)^2 \\ &= 1 - p(z-1) + p(z-1)Q_\infty + a(1 + Q_\infty^2 - 2Q_\infty) \end{aligned} \tag{7.5}$$

and rearranging

$$\begin{aligned} aQ_\infty^2 &+ \{\overbrace{p(z-1)-1}^{b} - 2a\}Q_\infty + a + \overbrace{1 - p(z-1)}^{-b} = 0 \Leftrightarrow \\ aQ_\infty^2 &+ (b-2a)Q_\infty + a - b = 0 \Leftrightarrow \\ Q_\infty &= \frac{2a - b \pm \sqrt{(b-2a)^2 - 4a(a-b)}}{2a} = \frac{2a - b \pm \sqrt{b^2}}{2a}. \end{aligned} \tag{7.6}$$

As $b > 0$ since $p$ eventually is larger that $\frac{1}{z-1}$ we find

$$Q_\infty = \begin{cases} 1 & \text{for } p \le p_c \\ \frac{a-b}{a} & \text{for } p > p_c. \end{cases}$$

The solution $Q_\infty = 1 \Rightarrow P_\infty(p) = 0$ belongs to the regime $p \le p_c$. The other solution is nontrivial and belongs to the regime $p > p_c$, and

$$Q_\infty = \frac{a-b}{a} = 1 - \frac{2p(z-1)-2}{p(z-1)(z-2)}.$$

(iii) The relevant solution has $Q_\infty < 1$. Substituting into the Equation (7.1)

$$\begin{aligned} P_\infty(p) &= p(1 - Q_\infty^z) \\ &= p\left[1 - \left(1 - \frac{2p(z-1)-2}{p(z-1)(z-2)}\right)^z\right] \\ &= p\left[1 - \left(1 - \frac{b}{a}\right)^z\right] \\ &= p - p(1 - \frac{b}{a})^z. \end{aligned} \tag{7.7}$$

Note that the ratio

$$\frac{b}{a} = \frac{p(z-1)-1}{p\frac{1}{2}(z-1)(z-2)} \to 0 \qquad \text{for } p \to \tfrac{1}{z-1} = p_c,$$

so $\frac{b}{a}$ is a small quantity for $p \to p_c$. Let $g(x) = (1-x)^z$. Taylor expanding to first order we find $g(0) = 1, g^{(1)}(x) = -z(1-x)^{z-1}, g^{(1)}(0) = -z$ so $(1-x)^z \approx 1 - zx$ for $x \to 0$. Thus

$$\begin{aligned} P_\infty(p) &= p - p(1 - \frac{b}{a})^z \\ &\approx p - p(1 - z\frac{b}{a}) \quad \text{for } p \to p_c \\ &= pz\frac{b}{a} \\ &= pz\frac{2p(z-1)-2}{p(z-1)(z-2)} \\ &= \frac{2z}{z-2}\left(p - \frac{1}{z-1}\right) \\ &= \frac{2z}{z-2}(p - p_c) \end{aligned} \tag{7.8}$$

with

$$p_c = \frac{1}{z-1}$$

and

$$A = \frac{2z}{z-2} \overset{z=3}{=} 6.$$

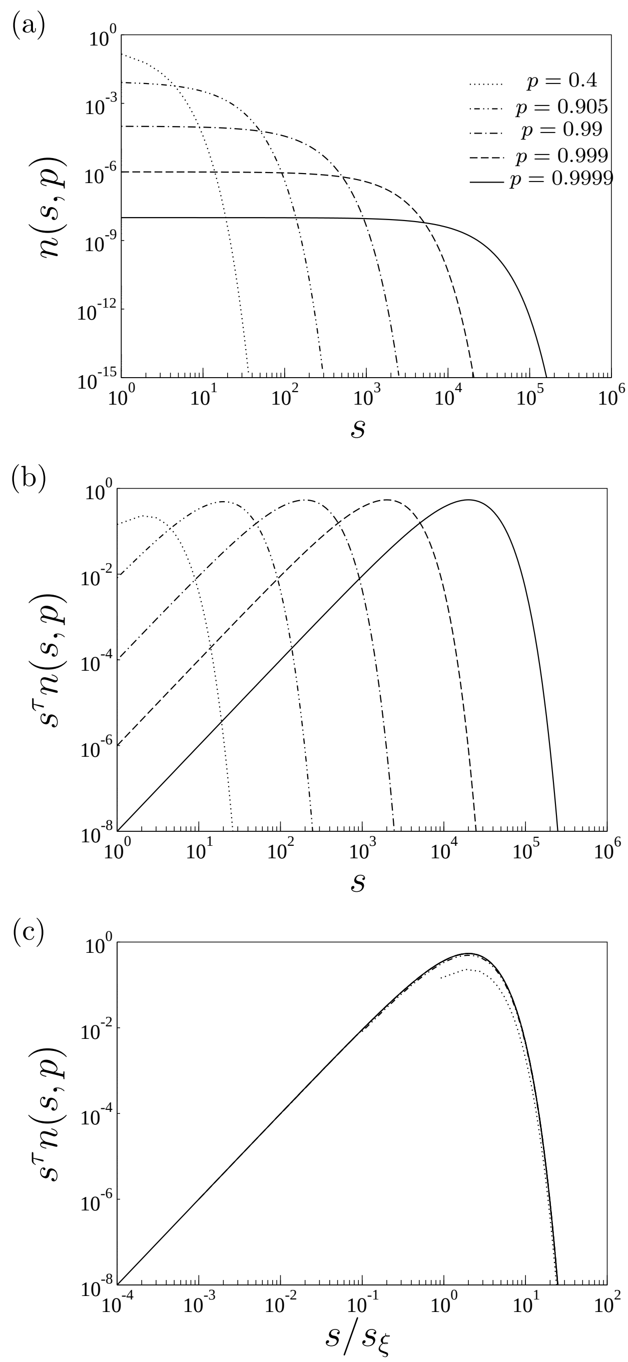


Figure 5.1: (a) The $d = 1$ cluster number density, $n(s, p)$, as a function of cluster size, $s$. The five curves correspond to $p = 0.4, 0.905, 0.99, 0.999, 0.9999$, respectively. (b) The transformed cluster number densities $s^2 n(s, p)$ versus the cluster size $s$. (c) For each transformed cluster number density $s^\tau n(s, p)$, the argument is rescaled from $s$ to $s/s_\xi$ where the characteristic cluster size, $s_\xi = -1/\ln(p) \approx 1, 10, 100, 1000, 10000$, respectively.

**5.** (i) See Figure 5.1 above.

(ii) The cluster number density in $d = 1$ can be written on the form

$$n(s, p) = s^{-2} \mathcal{G}_{1\mathrm{d}}(s/s_\xi) \tag{5.1}$$

where the function

$$\mathcal{G}_{1\mathrm{d}}(s/s_\xi) = (s/s_\xi)^2 \exp(-s/s_\xi). \tag{5.2}$$

Thus the transformed cluster number density

$$s^2 n(s,p) = \mathcal{G}_{1\mathrm{d}}\left(s/s_\xi\right) \tag{5.3}$$

is only a function of the ratio $s/s_\xi$ and by plotting $s^2 n(s,p)$ versus $s/s_\xi$, all the graphs collapse onto one curve, the graph of the function $\mathcal{G}_{1\mathrm{d}}$. Note that

$$\begin{aligned} \mathcal{G}_{1\mathrm{d}}\left(s/s_\xi\right) &= \left(s/s_\xi\right)^2 \exp(-s/s_\xi) \\ &= \begin{cases} \left(s/s_\xi\right)^2 & \text{for } s/s_\xi \ll 1 \\ \text{decays rapidly} & \text{for } s/s_\xi \gg 1, \end{cases} \end{aligned} \tag{5.4}$$

so for small arguments $s \ll s_\xi$, the function $\mathcal{G}_{1\mathrm{d}}$ increases as the argument squared while it decays exponentially fast for large arguments $s \gg s_\xi$, see Figure 5.1(c).