2.4 *Critical exponents inequality.*

Given the thermodynamic relation

$$\chi\,(C_H - C_M) = T\left(\frac{\partial\langle M\rangle}{\partial T}\right)_H^2 \tag{2.4.1}$$

As $C_M \geq 0$ and $\chi \geq 0$ it follows that

$$\chi\,C_H \geq T\left(\frac{\partial\langle M\rangle}{\partial T}\right)_H^2 . \tag{2.4.2}$$

Using the scaling of the different quantities close to the critical point

$$\begin{aligned}
\chi &\propto |T - T_c|^{-\gamma} && \text{for } T \to T_c,\\
C_H &\propto |T - T_c|^{-\alpha} && \text{for } T \to T_c,\\
\langle M\rangle &\propto (T_c - T)^{\beta} && \text{for } T \to T_c^- \text{ implying,}\\
\frac{\partial\langle M\rangle}{\partial T} &\propto -(T_c - T)^{\beta-1} && \text{for } T \to T_c^-
\end{aligned}$$

so by substituting into Equation (2.4.2) we find

$$\begin{aligned}
(T_c - T)^{-\gamma}\;(T_c - T)^{-\alpha} &\geq T_c\,\left(-(T_c - T)^{\beta-1}\right)^2 && \text{for } T \to T_c^-\\
(T_c - T)^{-\gamma-\alpha} &\geq T_c\,(T_c - T)^{2\beta-2} && \text{for } T \to T_c^-
\end{aligned}$$

from which we can conclude that

$$\begin{aligned}
-\gamma - \alpha \leq 2\beta - 2 &\Leftrightarrow\\
\gamma + \alpha \geq 2 - 2\beta &\Leftrightarrow\\
\alpha + 2\beta + \gamma \geq 2.&
\end{aligned} \tag{2.4.3}$$

Notice that the inequality actually hold as an *equality* for $d = 1, 2, 3,$ and 4 and the mean-field exponents for the Ising Model.

2.5 *The spin-spin correlation function and scaling relations.*

(i) The spin-spin correlation function

$$\begin{aligned} g(\mathbf{r}_i, \mathbf{r}_j) &= \langle\ (s_i - \langle s_i \rangle)\ (s_j - \langle s_j \rangle)\ \rangle \\ &= \langle\ s_i s_j - \langle s_i \rangle s_j - s_i \langle s_j \rangle + \langle s_i \rangle \langle s_j \rangle\ \rangle \\ &= \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle - \langle s_i \rangle \langle s_j \rangle + \langle s_i \rangle \langle s_j \rangle \\ &= \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle, \end{aligned} \tag{2.5.1}$$

where we use that the ensemble average operation $\langle \cdot \rangle$ is a linear operation and that the ensemble average of a constant is the constant itself.

(ii) Assuming that the system is translationally invariant, we substitute $m = \langle s_i \rangle = \langle s_j \rangle$ and find

$$\begin{aligned} g(\mathbf{r}_i, \mathbf{r}_j) &= \langle s_i s_j \rangle - m^2 \\ &= \langle s_j s_i \rangle - m^2 \\ &= g(\mathbf{r}_j, \mathbf{r}_i) \end{aligned} \tag{2.5.2}$$

from which is follows that the correlation function is symmetric and thus it is function of the relative distance between the spins at positions $\mathbf{r}_i$ and $\mathbf{r}_j$ only, that is,

$$g(\mathbf{r}_i, \mathbf{r}_j) = g(|\mathbf{r}_i - \mathbf{r}_j|). \tag{2.5.3}$$

(iii) (a) When $|\mathbf{r}_i - \mathbf{r}_j| \to \infty$, the spins become uncorrelated, assuming that we are not at the critical point that is! Thus

$$\begin{aligned} g(\mathbf{r}_i, \mathbf{r}_j) &= \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \\ &\to \langle s_i \rangle \langle s_j \rangle - \langle s_i \rangle \langle s_j \rangle \quad \text{for } |\mathbf{r}_i - \mathbf{r}_j| \to \infty \\ &= 0. \end{aligned} \tag{2.5.4}$$

(b) By definition the spin-spin correlation function of spin i with itself

$$g(\mathbf{r}_i, \mathbf{r}_i) = \langle s_i s_i \rangle - \langle s_i \rangle \langle s_i \rangle = \langle s_i^2 \rangle - \langle s_i \rangle^2. \tag{2.5.5}$$

Because $s_i = \pm 1 \Leftrightarrow s_i^2 = 1$ we have $\langle s_i^2 \rangle = \langle 1 \rangle = 1$. Also $\langle s_i \rangle = m$, so

$$g(\mathbf{r}_i, \mathbf{r}_i) = 1 - m^2. \tag{2.5.6}$$

We assume the external magnetic field $H = 0$ so we can replace m with $m_0(T)$. If $T \geq T_c$, the magnetisation $m_0 = 0$ so that

$$g(\mathbf{r}_i, \mathbf{r}_i) = \begin{cases} 1 & \text{for } T \geq T_c \\ 1 - m_0^2(T) & \text{for } T < T_c. \end{cases} \tag{2.5.7}$$

The zero-field magnetisation per spin $m_0(T) \to \pm 1$ for $T \to 0$, implying

$$g(\mathbf{r}_i, \mathbf{r}_i) \to 0 \quad \text{for } T \to 0. \tag{2.5.8}$$

This result emphasises that the correlation function measures the fluctuations of the spins *away* from the average magnetisation as is clear from the original definition

$$g(\mathbf{r}_i, \mathbf{r}_i) = \langle\, (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle)\, \rangle. \tag{2.5.9}$$

(c) In the limit $J/k_BT \ll 1$ (high temperatures relative to the coupling constant), the spins will be orientated randomly, that is, there is no correlations between the spins, so we expect $g(\mathbf{r}_i, \mathbf{r}_j) \to 0$.
In the limit $J/k_BT \gg 1$ (low temperatures relative to the coupling constant), the spins will be aligned, that is, there is no fluctuations away from the average spin, so we expect $g(\mathbf{r}_i, \mathbf{r}_j) \to 0$.

(iv) Because the susceptibility per spin diverges at the critical temperature

$$\chi(T, 0) \propto |T - T_c|^{-\gamma} \quad \text{for } T \to T_c \tag{2.5.10}$$

the volume integral of the correlation function must also diverge at the critical temperature,

$$\int_V g(\mathbf{r}) d^d\mathbf{r} \propto \int_a^\infty g(r) r^{d-1}\, dr \to \infty \quad \text{for } T \to T_c, \tag{2.5.11}$$

where a is a lower cutoff = lattice constant. This implies that $g(r)$ cannot decay exponentially with distance r at the critical point $T = T_c$ since this would make the integral convergent in the upper limit. However, the divergence is consistent with an algebraic decay. Assuming

$$g(r) \propto r^{-(d-2+\eta)} \quad \text{for } T = T_c, \text{ all } r = |\mathbf{r}| \tag{2.5.12}$$

then

$$\begin{aligned} \int_V g(\mathbf{r}) d^d\mathbf{r} &\propto \int_a^\infty g(r) r^{d-1}\, dr \\ &\propto \int_a^\infty r^{-(d-2+\eta)} r^{d-1}\, dr \\ &= \int_a^\infty r^{1-\eta}\, dr \\ &= \begin{cases} \left[\frac{1}{2-\eta} r^{2-\eta}\right]_a^\infty & \text{if } \eta \neq 2 \\ \left[\ln(r)\right]_a^\infty & \text{if } \eta = 2 \end{cases} \end{aligned}$$

that is, the integral will only diverge if the critical exponent $\eta \leq 2$. The divergence is logarithmic if $\eta = 2$ and algebraically if $\eta < 2$.

(v) (a) The correlation length diverges as $\xi(T, 0) \propto |T_c - T|^{-\nu}$ for $T \to T_c$. The critical exponent ν is independent on whether T_c is approached from below or above, however, the amplitude might differ, as in the graph below. For $T > T_c$, the correlation length sets the upper linear distance over which spins are correlated. It is also identified as the linear size of the typical (characteristic) largest cluster of aligned spins and measures the typical largest fluctuation away from states with randomly oriented spin. For $T < T_c$, the correlation length measures the fluctuations away from the fully ordered state, that is, the upper linear size of the holes in the percolating cluster of aligned spins. There will be holes on all scales up to the correlation length.

(b) When $T \neq T_c$ a finite correlation length ξ is introduced and

$$g(|\mathbf{r}|) \propto r^{-(d-2+\eta)} \mathcal{G}_\pm(r/\xi) \quad \text{for } T \to T_c, \tag{2.5.13}$$

where

$$\xi \propto |T_c - T|^{-\nu} \quad \text{for } T \to T_c. \tag{2.5.14}$$

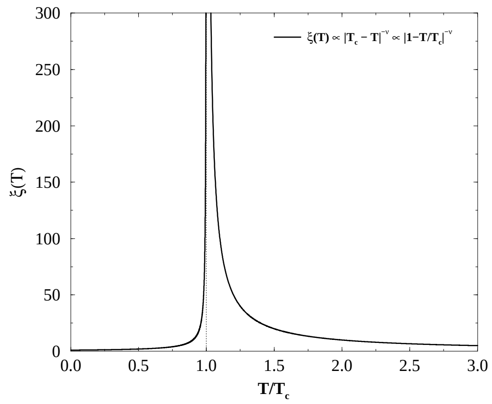

Fig. 2.5.1 The correlation length $\xi(T,0)$ as a function of the temperature T in units of the critical temperature T_c.

Consider the relation between the susceptibility per spin and the correlation function

$$k_B T\chi \propto \int_V g(\mathbf{r}) d^d\mathbf{r}. \tag{2.5.15}$$

The left-hand side (LHS):

$$k_B T\chi \propto |T - T_c|^{-\gamma} \quad \text{for } T \to T_c. \tag{2.5.16}$$

The right-hand side (RHS):

$$\begin{aligned}\int_V g(\mathbf{r}) d^d\mathbf{r} &\propto \int_0^\infty r^{-(d-2+\eta)}\mathcal{G}_\pm(r/\xi) r^{d-1}\, dr \\ &= \int_0^\infty r^{1-\eta}\mathcal{G}_\pm(r/\xi)\, dr \\ &= \int_0^\infty (\tilde{r}\xi)^{1-\eta}\mathcal{G}_\pm(\tilde{r})\, d\tilde{r}\xi \quad \text{with } r = \tilde{r}\xi \\ &= \xi^{2-\eta}\int_0^\infty \tilde{r}^{1-\eta}\mathcal{G}_\pm(\tilde{r})\, d\tilde{r} \\ &= |T - T_c|^{-\nu(2-\eta)}\int_0^\infty \tilde{r}^{1-\eta}\mathcal{G}_\pm(\tilde{r})\, d\tilde{r} \quad \text{for } T \to T_c^\pm.\end{aligned} \tag{2.5.17}$$

The integral is just a number (which numerical value, however, depends on from which side T_c is approached due to the two different scaling functions $\mathcal{G}_\pm$), so we can conclude by comparing the LHS with the RHS that

$$\gamma = \nu(2 - \eta). \tag{2.5.18}$$

(c) We assume $T \leq T_c$ and consider the situation in zero external field $H = 0$ with m_0 replacing m. We define

$$\tilde{g}(r) = g(r) + m_0^2 = \langle s_i s_j \rangle. \tag{2.5.19}$$

For $T < T_c$, the correlation length $\xi < \infty$. As the correlation length sets the upper limit of the linear scale over which spins are correlated, the spins will be uncorrelated in the limit $r \to \infty$ as $r \gg \xi$. Thus

$$\tilde{g}(r) = \langle s_i s_j \rangle \to \langle s_i \rangle \langle s_j \rangle = m_0^2 \propto (T_c - T)^{2\beta} \quad \text{for } T \to T_c^-. \tag{2.5.20}$$

At $T = T_c$ where the correlation length in infinite, the magnetisation is zero in zero external field, i.e., $m_0(T_c) = 0$. Thus

$$\tilde{g}(r) = g(r) \propto r^{-(d-2+\eta)} \quad \text{at } T = T_c. \tag{2.5.21}$$

One would thus expect, ala finite-size scaling in percolation theory that

$$\tilde{g}(r) \propto \begin{cases} r^{-(d-2+\eta)} & \text{for } r \ll \xi \\ \xi^{-(d-2+\eta)} & \text{for } r \gg \xi. \end{cases} \tag{2.5.22}$$

This is the reason for considering the function $\tilde{g}(r)$ and not $g(r)$ as the latter will approach zero for $r \gg \xi$. Thus for $T < T_c$ where the correlation length is finite, we expect

$$\tilde{g}(r) \propto \xi^{-(d-2+\eta)} \propto |T - T_c|^{\nu(d-2+\eta)} \quad \text{for } r \gg \xi \tag{2.5.23}$$

implying the scaling relation

$$2\beta = \nu(d - 2 + \eta) \Leftrightarrow d - 2 + \eta = 2\beta/\nu. \tag{2.5.24}$$

2.6 *Diluted Ising model.*

(i) A spin is situated on each lattice site. However, the spin only interact with with the nearest neighbours with probability p. Identifying a nonzero coupling constant $J_{ij} = J > 0$ as an occupied bond and $J_{ij} = 0$ as an empty bond, we have an exact mapping onto a bond percolation theory problem.

(ii) (a) In order to minimise the energy, all spins within a percolation cluster will have all spins pointing in the same direction. However, spins belonging to different percolation clusters are not correlated.

(b) Within a cluster, $s_i = s_j$ so $s_i s_j = s_i^2 = 1$ implying $\langle s_i s_j \rangle = 1$ if the spins belong to the same cluster. If the spins i and j belong to different clusters, they are not correlated at all, that is, given, e.g., that $s_i = 1$ then $s_j = 1$ with probability 0.5 and $s_j = -1$ with probability 0.5 leaving $\langle s_i s_j \rangle = 0$. Hence

$$\langle s_i s_j \rangle = \begin{cases} 1 & i, j \text{ in the same percolation cluster} \\ 0 & \text{otherwise.} \end{cases} \tag{2.6.1}$$

(c) For $p < p_c$ all clusters are finite. Since the clusters are not correlated, the average magnetisation is zero. For $p > p_c$, we can argue, that all the finite clusters do not contribute to the magnetisation which then becomes equal to $P_\infty(p)$, the density of the infinite cluster. The orientation of the infinite cluster is random (in zero external field). Since $P_\infty(p) = 0$ for $p < p_c$

$$m_0(p) = \pm P_\infty(p) \tag{2.6.2}$$

(iii) (a) $P_\infty(p)$ is the probability for a spin to belong to the infinite cluster. As $\tanh(sH/k_BT) \to 0$ for $H \to 0$, the last term will vanish and

$$m_0(p) = \lim_{H \to 0} m(p, H) = \pm P_\infty(p)$$

consistent with the result of (ii)(c).

(b) The susceptibility in zero external field

$$\chi = \left(\frac{\partial m}{\partial H} \right)_{H=0}.$$

Assuming $H \ll k_B T$ we use the expansion $\tanh(sH/k_BT) \approx sH/k_BT + \mathcal{O}\left((sH/k_BT)^3\right)$. Since $P_\infty(p)$ does not depend on the external field, we find,

$$\chi = \left(\frac{\partial m}{\partial H}\right)_{H=0} = \sum_{s=1}^{\infty} \frac{s^2 n(s,p)}{k_B T} \propto \chi(p) \propto |p - p_c|^{-\gamma}$$

as the divergence of the second moment of the cluster size density $n(s,p)$ is characterized by the exponent γ when $p \to p_c$.

(iv) When $p < p_c$, the magnetisation in zero external field $m_0(p) = 0$. Within a cluster $\langle s_i s_j \rangle = 1$. In a cluster of size s there are a total of s^2 different pairs, so $\frac{1}{k_BT}\sum_i \sum_j \langle s_i s_j \rangle = \frac{1}{k_BT} s^2$. We can calculate the average susceptibility by summing over all possible cluster sizes weighted by the cluster size distribution, that is,

$$\chi = \sum_{s=1}^{\infty} \left(\frac{1}{k_BT} \sum_i \sum_j \langle s_i s_j \rangle\right) n(s,p) = \frac{1}{k_BT} \sum_{s=1}^{\infty} s^2 n(s,p).$$