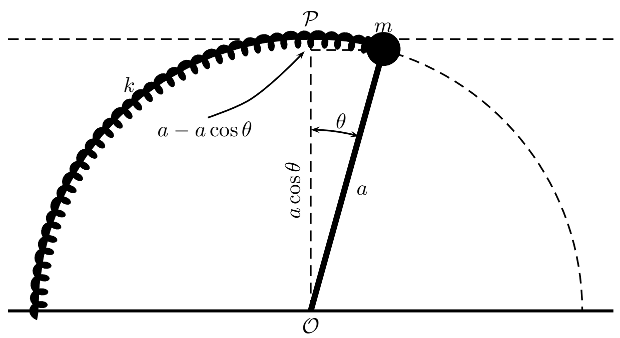
2.7 *Second-order phase transition in a mass-spring system: Landau theory.*

(i) The total energy of the mass-spring system

$$\begin{aligned} U(\theta) &= \text{elastic potential energy} + \text{gravitational potential energy} \\ &= \frac{1}{2}k(a\theta)^2 + mg(a\cos\theta - a) \\ &= \frac{1}{2}ka^2\theta^2 + mga(\cos\theta - 1) \end{aligned}$$



(ii) (a) We expand the cosine to fourth order to find

$$\begin{aligned} U(\theta) &= \frac{1}{2}ka^2\theta^2 + mga(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots - 1) \\ &= \frac{a}{2}(ka - mg)\theta^2 + \frac{mga}{24}\theta^4 + \mathcal{O}(\theta^6) \end{aligned}$$

where the coefficient of the fourth-order term is positive while the coefficient of the second-order term is zero for $ka = mg$ and changes sign from positive when $ka > mg$ to negative when $ka < mg$.

(b) As the total energy $U(\theta)$ is an even function in θ (reflecting the symmetry of the problem), all the odd terms in the Taylor expansion around $\theta = 0$ are zero.

(c) When $ka > mg$, the unique minimum is at $\theta_0 = 0$. When $ka = mg$, the unique minimum is at $\theta_0 = 0$. When $ka < mg$, there are two minima at $\pm\theta_0 \neq 0$.

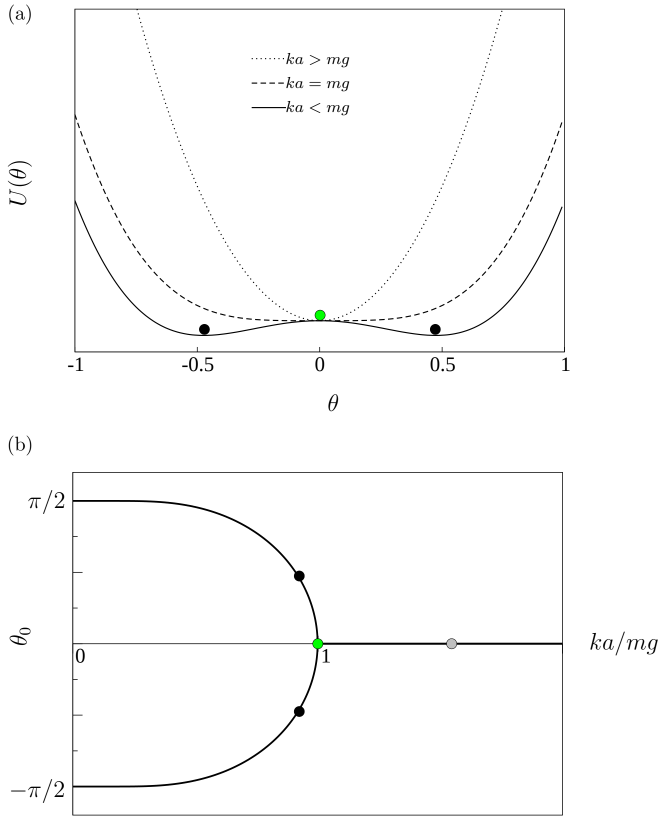


Fig. 2.7.1 (a) The energy, $U(\theta)$, versus the angle θ. The solid circles show the position of the minima of the energy of the corresponding graph. For $ka > mg$, the minimal energy implies $\theta = 0$. For $ka = mg$, the trivial solution $\theta = 0$ is marginally stable However, for $ka < mg$, the minimal energy implies $\theta = \pm\theta_0 \neq 0$. (b) The angle of equilibrium, θ_0 as a function of the ratio ka/mg.

(d) The system is in equilibrium when $dU/d\theta = 0$. Hence

$$\begin{aligned}\frac{dU}{d\theta} &= a(ka - mg)\theta + \frac{mga}{6}\theta^3 \\ &= mga\theta\left(\frac{ka}{mg} - 1 + \frac{1}{6}\theta^2\right) \\ &= 0 \end{aligned} \qquad (2.7.1)$$

with solutions

$$\theta_0 = \begin{cases} 0 & \text{for } \frac{ka}{mg} \geq 1 \\ \pm\sqrt{6(1 - ka/mg)} & \text{for } \frac{ka}{mg} < 1 \end{cases}$$
$$= \begin{cases} 0 & \text{for } \frac{m_c}{m} \geq 1 \\ \pm\sqrt{6[(m - m_c)/m]} & \text{for } \frac{m_c}{m} < 1, \end{cases}$$

where $m_c = ka/g$.

(e) See previous Figure.

(f) Landau suggested a simplistic general theory of second-order phase transitions based on expanding the free energy in powers of the order parameter. In the absence of a magnetic-like field, symmetry dictates that only even powers of the order parameter appear in the expansion. For example, in the Ising model

$$f - f_0 = a_2(T - T_c)m^2 + a_4 m^4 \quad \text{with } a_2, a_4 > 0,$$

where an expansion up to fourth order is sufficient to give a qualitative description of second-order phase transitions occurring at temperature T_c. The term f_0 is an unimportant constant, while $a_4 > 0$ in order for the free energy to be physically realistic, i.e. not minimised by extreme values of the order parameter.

As written, the left-hand side is given by a quartic polynomial which always has one trivial solution, $m = 0$, and two non-trivial solutions, $m = \pm m_0(T)$, so long as $T < T_c$. As T passes through T_c from above, the trivial solution becomes unstable and two stable non-trivial solutions appear. Below T_c, therefore, the order parameter of the system is non-zero.

(g) The order parameter of the mass-spring system is the equilibrium angle θ_0 which is zero for $m \leq m_c$ and non-zero for $m > m_c$. The critical value of the variable mass $m_c = ka/g$.

2.8 *Scaling ansatz of free energy and scaling relations.*

Consider the Ising model on a d-dimensional lattice in an external field H.

(i) (a) The total energy for a system of N spins $s_i = \pm 1$ with constant nearest-neighbour interactions $J > 0$ placed in a uniform external field H is

$$E_{\{s_i\}} = -J \sum_{\langle ij \rangle} s_i s_j - H \sum_{i=1}^{N} s_i,$$

where the notation $\langle ij \rangle$ restricts the sum to run over all distinct nearest-neighbour pairs.

(b) Let $M_{\{s_i\}} = \sum_{i=1}^{N} s_i$ denote the total magnetisation and $\langle M \rangle$ the average total magnetisation. The order parameter for the Ising model is defined as the magnetisation per spin

$$m(T, H) = \lim_{N \to \infty} \frac{\langle M \rangle}{N}.$$

Consider the free energy $F = \langle E \rangle - TS$. The ratio of the average total energy, $\langle E \rangle$, to the temperature times entropy, TS, defines a dimensionless scale $J/k_B T$. A competition exists between the tendency to randomise the orientation of spins for $J \ll k_B T$, and a tendency to align spins for $J \gg k_B T$. In the former case, the free energy is minimised by maximising the entropic term: the magnetisation is zero because the spins point up and down randomly. In the latter case, the free energy is minimised by minimising the total energy: the magnetisation is non-zero because the spins tend to align. Since the entropy in the free energy is multiplied by temperature, for sufficiently low temperatures, the minimisation of the free energy is dominated by the minimisation of the total energy. Therefore, at least qualitatively, there is a possibility of a phase transition from a phase with zero magnetisation at relatively high temperatures, to a phase with non-zero magnetisation at relatively low temperatures.

We assume that the singular part of free energy per spin is a generalised homogeneous function,

$$f(t, h) = b^{-d} f(b^{y_t} t, b^{y_h} h) \quad \text{for } t \to 0^{\pm}, h \to 0, b > 0. \qquad (2.8.1)$$

(ii) (a) The critical exponent α associated with the specific heat in zero external field is defined by

$$c(t,0) \propto |t|^{-\alpha} \quad \text{for } t \to 0.$$

The specific heat is related to the free energy per spin:

$$c(t,h) \propto \left(\frac{\partial^2 f}{\partial t^2}\right) \propto b^{2y_t - d} f''(b^{y_t} t, b^{y_h} h)$$

Choosing $b = |t|^{-1/y_t}$ and setting $h = 0$ we find

$$c(t,0) \propto |t|^{-\frac{2y_t - d}{y_t}} f''(\pm 1, 0) \quad \text{for } t \to 0^{\pm},$$

and we identify

$$\alpha = \frac{2y_t - d}{y_t}$$

(b) The critical exponent β associated with the order parameter (magnetisation per spin) in zero external field is defined by

$$m(t,0) \propto |t|^{\beta} \quad \text{for } t \to 0^{-}.$$

The magnetisation per spin is related to the free energy per spin:

$$m(t,h) \propto -\left(\frac{\partial f}{\partial h}\right) \propto b^{y_h - d} f'(b^{y_t} t, b^{y_h} h).$$

Choosing $b = |t|^{-1/y_t}$ and setting $h = 0$ we find

$$m(t,0) \propto |t|^{\frac{d - y_h}{y_t}} f'(\pm 1, 0) \quad \text{for } t \to 0^{\pm},$$

and we identify

$$\beta = \frac{d - y_h}{y_t}$$

(c) The critical exponent γ associated with the susceptibility in zero external field is defined by

$$\chi(t,0) \propto |t|^{-\gamma} \quad \text{for } t \to 0.$$

The susceptibility is related to the free energy per spin:

$$\chi(t,h) \propto -\left(\frac{\partial^2 f}{\partial h^2}\right) \propto b^{2y_h-d} f''(b^{y_t}t, b^{y_h}h).$$

Choosing $b = |t|^{-1/y_t}$ and setting $h = 0$ we find

$$\chi(t,0) \propto |t|^{-\frac{2y_h-d}{y_t}} \quad \text{for } t \to 0$$

and we identify

$$\gamma = \frac{2y_h - d}{y_t}.$$

(d) The critical exponent δ associated with the order parameter in the critical temperature is defined by

$$m(0,h) \propto \text{sign}(h)|h|^{1/\delta} \quad \text{for } h \to 0.$$

The magnetisation per spin is related to the free energy per spin:

$$m(t,h) \propto -\left(\frac{\partial f}{\partial h}\right) \propto b^{y_h-d} f'(b^{y_t}t, b^{y_h}h).$$

Choosing $b = |h|^{-1/y_h}$ and setting $t = 0$ we find

$$m(0,h) \propto |h|^{\frac{d-y_h}{y_h}} \quad \text{for } h \to 0$$

and we identify

$$\delta = \frac{y_h}{d - y_h}$$

(e) We find

$$\begin{aligned}\alpha + 2\beta + \gamma &= \frac{2y_t - d + 2d - 2y_h + 2y_h - d}{y_t} \\ &= 2\end{aligned}$$

and

$$\begin{aligned}\beta(\delta-1) &= \frac{d-y_h}{y_t}\left(\frac{y_h}{d-y_h}-1\right)\\ &= \frac{d-y_h}{y_t}\left(\frac{2y_h-d}{d-y_h}\right)\\ &= \frac{2y_h-d}{y_t}\\ &= \gamma.\end{aligned}$$