Answers to Additional Exercises

Statistical Mechanics 2006-2007

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1. *Taylor expansion*

(i) The Taylor expansion of the function $f(x) = \ln(1-x)$ about the point $x = 0$ to order 3 is given by

$$f(x) \approx= f(0) + f^{(1)}(0)x + \frac{f^{(2)}(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3. \tag{1.1}$$

We find

$$\begin{aligned} f(0) &= \ln(1) = 0 \\ f^{(1)}(x) &= \frac{1}{1-x} \cdot (-1) = -(1-x)^{-1} \quad \Rightarrow \quad f^{(1)}(0) = -1, \\ f^{(2)}(x) &= -(1-x)^{-2} \quad \Rightarrow \quad f^{(2)}(0) = -1, \\ f^{(3)}(x) &= -2(1-x)^{-3} \quad \Rightarrow \quad f^{(3)}(0) = -2, \end{aligned}$$

that is,

$$\begin{aligned} f(x) &\approx 0 - x + \frac{-1}{2!}x^2 + \frac{-2}{3!}x^3 = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \\ &\to -x \quad \text{for} \quad x \to 0. \end{aligned}$$

Graphically,

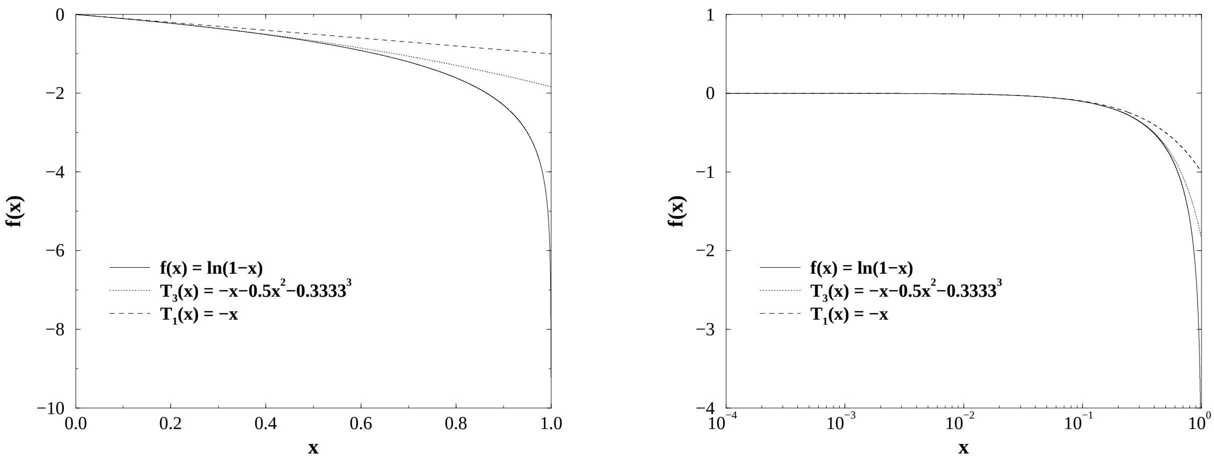

Figure 1.1: The Taylor expansion of the function $\ln(1-x)$ (solid line) to order 3 (dotted line) and order 1 (dashed line). When $x \to 0$, the approximation $\ln(1-x) \approx -x$ is valid. Notice that in (a), the x-axis is linear while in (b) it is logarithmic. We use the notation $T_n(x)$ for the Taylor expansion to order n.

(ii) The Taylor expansion of the function $f(p) = p - \frac{(1-p)^3}{p^2}$ about the point $p = \frac{1}{2}$ to order 2 is given by

$$f(p) \approx f(\frac{1}{2}) + f^{(1)}(\frac{1}{2})(p - 1/2) + \frac{f^{(2)}(\frac{1}{2})}{2!}(p - 1/2)^2. \tag{1.2}$$

We find

$$\begin{aligned} f(\frac{1}{2}) &= \frac{1}{2} - \frac{(\frac{1}{2})^3}{(\frac{1}{2})^2} = 0 \\ f^{(1)}(p) &= 1 + 3(1-p)^2 p^{-2} + 2(1-p)^3 p^{-3} \quad \Rightarrow \quad f^{(1)}(\frac{1}{2}) = 6, \\ f^{(2)}(p) &= -6(1-p)p^{-2} - 6(1-p)^2 p^{-3} - 6(1-p)^2 p^{-3} - 6(1-p)^3 p^{-4} \quad \Rightarrow \quad f^{(2)}(\frac{1}{2}) = -48, \end{aligned}$$

that is, to second order in $(p - 1/2)$

$$f(p) \approx 6(p - 1/2) - 24(p - 1/2)^2,$$

implying $A = 6$ and $B = -24$.

Graphically,

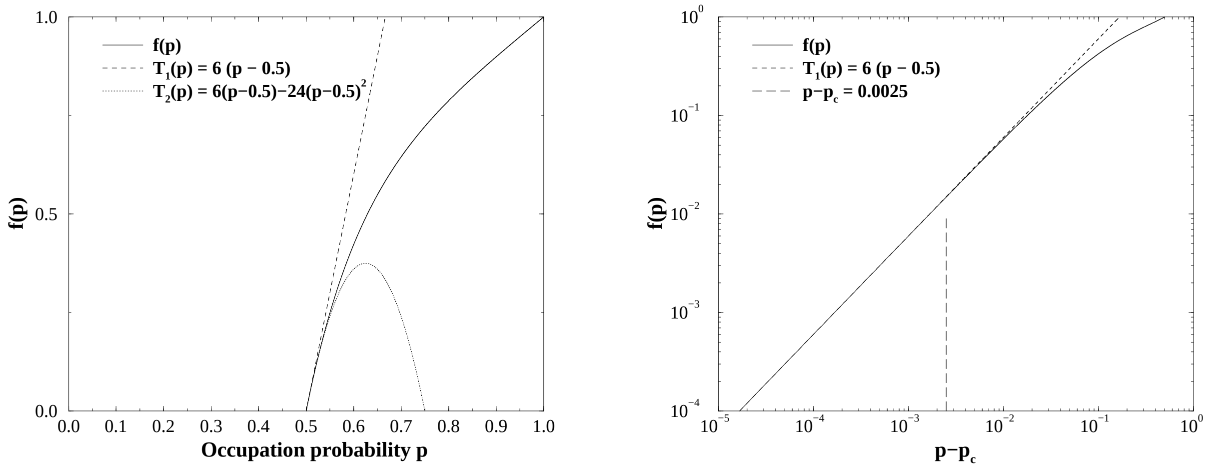

Figure 1.2: (a) The Taylor expansion of the function $f(p)$ (solid line) to first order (dashed line) and second order (dotted line). (b) In a double logarithmic plot, one can see that when $p - 1/2 \leq 0.0025$, the first order approximation $f(p) \approx 6(p - \frac{1}{2})$ is excellent and similarly for the Taylor expansion of second order. Again we use the common notation $T_n(p)$ for the Taylor expansion to order n.

2. *Power-law probability density*

(i) As $P(h) = 0$ for $h < h_{min}$, the condition for normalisation is

$$\int_{h_{min}}^{\infty} P(h)\, dh = \int_{h_{min}}^{\infty} Ah^{-2}\, dh = A[-h^{-1}]_{h_{min}}^{\infty} = Ah_{min}^{-1} = 1 \Leftrightarrow A = h_{min}.$$

(ii) (a) By definition, we have to integrate the probability density over $h \geq h_{max}$:

$$P(h \geq h_{max}) = \int_{h_{max}}^{\infty} h_{min} h^{-2}\, dh = h_{min}[-h^{-1}]_{h_{max}}^{\infty} = \frac{h_{min}}{h_{max}}.$$

(b) The average number of days one would have to wait to see one event with $h \geq h_{max}$ is

$$\frac{1}{P(h \geq h_{max})} = \frac{h_{max}}{h_{min}}.$$

(iii) (a) There is no upper limit to the level of the river, so it is impossible to guarantee safety forever.

(b) There are $365N$ days in N years. The probability of having no overflow in $365N$ consecutive days is

$$\begin{aligned} P(\text{No overflow in } 365N \text{ days}) &= \left(1 - \frac{h_{min}}{h_{max}}\right)^{365N} \geq p \Rightarrow \\ 1 - \frac{h_{min}}{h_{max}} &\geq p^{\frac{1}{365N}} \Rightarrow \\ h_{max} &\geq \frac{h_{min}}{1 - p^{\frac{1}{365N}}}. \end{aligned}$$

(c) Inserting $N = 10, p = 0.90$ and $h_{min} = 0.01$ m we find

$$h_{max} \geq \frac{0.01 \text{ m}}{1 - 0.90^{\frac{1}{3650}}} \approx 346 \text{ m}.$$

(iv) (a) The average level

$$\langle h \rangle = \int_{h_{min}}^{\infty} h_{min} h\, h^{-2}\, dh = h_{min} \int_{h_{min}}^{\infty} h^{-1}\, dh = h_{min}[\ln(h)]_{h_{min}}^{\infty} = \infty.$$

Note that this is a so-called marginal case where the average level diverges logarithmically. A power-law probability with an exponent less than -2 would have a finite average value, while a power-law probability with an exponent greater than -2 would diverge algebraically.

(b) One could imagine that there exists an upper cut-off, h_c, in the level of the river for the probability density such that $P(h) = 0$ for $h \geq h_c$. Another possibility would be to modify the power-law exponent such that it is slightly less than -2.