Universality Class of One-Dimensional Directed Sandpile Models

Matthew Stapleton* and Kim Christensen†
Blackett Laboratory, Imperial College London,
Prince Consort Road, London SW7 2BW, United Kingdom
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A general n-state directed 'sandpile' model is introduced. The stationary properties of the n-state model are derived for $n < \infty$, and analytical arguments based on a central limit theorem show that the model belongs to the universality class of the totally asymmetric Oslo model, with a crossover to uncorrelated branching process behavior for small system sizes. Hence, the central limit theorem implies the existence of a universality class of one-dimensional directed sandpile models.

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Sandpile models have attracted much analytical attention in recent years, largely due to their application to the development of self-organized criticality (SOC)[1–4]. Analytical solutions, scaling arguments and numerics have shown that many models share the same critical exponents and scaling functions, leading to the notion of universality classes for such systems, as in equilibrium systems [2, 5, 6]. In the following, we show that an n-state directed model may be analyzed in such a way that it becomes transparent that, for $n < \infty$, a central limit theorem for dependent random variables [7] is the origin of its universality. As the system size is increased, the model exhibits a crossover from uncorrelated branching process behavior to that of the totally asymmetric Oslo model (TAOM) recently solved in Ref. [8, 9]. The crossover is characterized by an n dependent crossover length, ξ_n.

The model considered is an n-state directed 'sandpile' model. The system exists on a one-dimensional lattice with L sites. Each site, i, is in one of n states, $z_i \in [0, n-1]$, which represents the number of particles on site i. We use the term 'charging site i' when adding a particle to site i and 'charging a site m times' when adding m particles to that site. 'Charging the system' specifically means adding a particle to site $i = 1$ of that system.

At the beginning of each time step, the system is charged once. Site $i = 1$ may then topple a number of times, each toppling redistributing a particle to the next site, $z_1 \to z_1 - 1$ and $z_2 \to z_2 + 1$. When site $i = 2$ receives a particle it is considered as having been charged and may also undergo topplings, redistributing particles to site $i = 3$, and so on, with particles being passed to sites of increasing i. Note that when site $i = L$ topples, the redistributed particle will leave the system. When all activity ceases, a new time step commences. The avalanche size, s, is defined as the total number of topplings during a single time step. The only restrictions on the toppling rules are: (i) A toppling may never cause z_i to become negative. (ii) If $z_i > n - 1$, then site i must topple. (iii) When site i topples, it charges site $i + 1$ only. (iv) Bulk conservation: Particles only leave the system at the boundary site $i = L$ and enter when the system is charged.

For the following general discussion, there is no need to specify the toppling rules in any more detail. Later, when presenting numerical results, we will consider a specific implementation of a general class of toppling rules that satisfies the 'restrictions' (i)-(iv) above.

The quantity of interest in sandpile models is the avalanche-size probability, $P(s; L, 1)$, which is the probability of observing an avalanche of size s in a system of size L that has been charged once. SOC is associated with an avalanche-size probability which obeys simple finite-size scaling

$$P(s; L, 1) = a s^{-\tau} \mathscr{G}\left(s/bL^D\right) \text{ for } s, L \gg 1, \quad (1)$$

where a and b are non-universal constants, τ and D are universal critical exponents, and $\mathscr{G}$ is a universal scaling function. The kth moment of the avalanche-size probability is

$$\begin{aligned} Q_{L,1}^{(k)} &= \sum_{s=1}^{\infty} s^k P(s; L, 1) \\ &\approx \int_1^{\infty} ds \; a s^{k-\tau} \mathscr{G}(s/bL^D) \\ &= a b^{1+k-\tau} G_k(L) L^{D(1+k-\tau)}, \end{aligned} \quad (2)$$

where the sum has been approximated by an integral and $G_k(L) = \int_{1/bL^D}^{\infty} du \; u^{k-\tau} \mathscr{G}(u)$. Provided that $0 < G_k(\infty) < \infty$, we have

$$Q_{L,1}^{(k)} = \Gamma_k L^{\gamma_k} \quad \text{for } L \gg 1, \quad (3)$$

*Electronic address: ms599@imperial.ac.uk
†Electronic address: k.christensen@imperial.ac.uk

where $\gamma_k = D(1+k-\tau)$ is a universal exponent and $\Gamma_k = ab^{1+k-\tau}G_k(L)$ is a non-universal amplitude which is a constant for $L \gg 1$. Hence, the scaling of the moments with system size L is a universal feature which is independent of particular details of the dynamics if Eq. (1) is valid, the approximation to the integral in Eq. (2) does not effect the scaling behavior of $Q^{(k)}_{L,1}$, and $G_k(L)$ approaches a non-zero constant for $L \to \infty$. These assumptions can be justified numerically and we will only consider cases when they are valid. We may then calculate the critical exponents τ and D from, e.g., γ_1 and γ_2 [12]. Therefore, our task is to determine the scaling of the first and second moments with system size, L.

Using a Markov matrix approach, similar to that introduced in Ref. [9], it can be shown that if the Markov matrix representing the evolution operator for the model is regular and $n < \infty$, then there is a unique stationary state where sites are occupied independently by z particles with probability p_z. Hence, we find that the probability of occurrence of a configuration $\{z_i\} = \{z_1, z_2, \ldots, z_L\}$ is

$$p_{\{z_i\}} = \prod_{i=1}^{L} p_{z_i}, \tag{4}$$

where the values of p_z depend on the details of the toppling rules. This is known as a product state and has the property that there are no spatial correlations. However, we shall shortly argue that there exists a crossover length, ξ_n, such that for system sizes $L \gg \xi_n$, there are temporal correlations which produce non-trivial behavior.

We define

$$Q^{(k)}_{L,m} \equiv \sum_{s=0}^{\infty} s^k P(s; L, m) \tag{5}$$

as the kth moment of the avalanche-size probability, $P(s; L, m)$, for a system of size L which has been charged m times.

The first moment is easily derived from the fact that, in the stationary state, the average number of particles which leave the system through the open boundary must equal the number of times the system is charged. Each of the m particles topples exactly L times, and

$$Q^{(1)}_{L,m} = mL, \tag{6}$$

implying $\gamma_1 \equiv D(2-\tau) = 1$.

To derive the scaling of the second moment, we introduce $P(t, s; 1, L, m)$ as the joint probability that a system of size $1 + L$ which has been charged m times undergoes t topplings in the first site and s in the remaining L sites. A recursion relation for the second moment may be derived by noting that since the model is directed and the stationary state is a product state,

$$P(t, s; 1, L, m) = P(t; 1, m) P(s; L, t). \tag{7}$$

Hence

$$\begin{aligned} Q^{(2)}_{L+1,m} &= \sum_{s,t=0}^{\infty} (t+s)^2 P(t; 1, m) P(s; L, t) \\ &= \sum_{s=0}^{\infty} P(s; 1, m) Q^{(2)}_{L,s} + (2L+1) Q^{(2)}_{1,m} \end{aligned} \tag{8}$$

where we have used $Q^{(1)}_{L,s} = sL$. Equation (8) may be interpreted as the master equation for a generalized random walker where $Q^{(2)}_{L,s}$ represents the density of random walkers at 'time' L and 'position' s, with hopping probabilities varying in space according to $P(s; 1, m)$. The term $(2L+1)Q^{(2)}_{1,m}$ is a source term which injects walkers into each point in space. Defining

$$\tilde{Q}^{(2)}_{L,m} \equiv Q^{(2)}_{L,m} - m^2 L^2 \tag{9}$$

and substituting into Eq. (8) we find

$$\tilde{Q}^{(2)}_{L+1,m} = \sum_{s=0}^{\infty} P(s; 1, m) \tilde{Q}^{(2)}_{L,s} + (L+1)^2 \tilde{Q}^{(2)}_{1,m}. \tag{10}$$

Equation (10) is exact. We can extract the scaling for $m = 1$ with system size L assuming $\tilde{Q}^{(2)}_{1,m}$ is independent of m.

First we show that $\tilde{Q}^{(2)}_{1,m}$ is finite for $n < \infty$. Using $Q^{(1)}_{1,m} = m$, we have from the definition in Eq. (9) that

$$\tilde{Q}^{(2)}_{1,m} = \sum_{s=0}^{\infty} (s-m)^2 P(s; 1, m). \tag{11}$$

Equation (11) is the width of the probability $P(s; 1, m)$, which is finite for $n < \infty$: Consider a site with z particles which is charged m times. After s topplings have taken place it will contain $z' = z + m - s$ particles. Since both z and z' must lie between 0 and $n-1$ there are bounds on s, $\max(0, m-n+1) = s_{\min} \le s \le s_{\max} = m + n - 1$, and the probability $P(s; 1, m)$ has non-zero values only for $s \in [s_{\min}, s_{\max}]$ and hence $\tilde{Q}^{(2)}_{1,m}$ is finite for $n < \infty$.

The restriction of the support of $P(s; 1, m)$ suggests further that $\tilde{Q}^{(2)}_{1,m} \approx \text{const}$. We make the assumption, to be justified later, that this is indeed true, and write $\tilde{Q}^{(2)}_{1,m} \equiv D$. Inserting this into Eq. (10), we find

$$\tilde{Q}^{(2)}_{L+1,m} = \sum_{s=0}^{\infty} P(s;1,m)\tilde{Q}^{(2)}_{L,s} + (L+1)^2 D. \quad (12)$$

The solution to Eq. (12) is simply a generalization of that obtained in Ref. [9],

$$\tilde{Q}^{(2)}_{L,m} = \sum_{l=1}^{L} D\, l^2 \sum_{k=1}^{\infty} \psi_{L-l,k}(m), \quad (13a)$$

$$\psi_{L+1,k}(m) = \sum_{s=0}^{\infty} P(s;1,m)\psi_{L,k}(s). \quad (13b)$$

Equation (13b) is also the master equation for a density of generalized random walkers, where L represents time, m position and the hopping rule varies in space according to $P(s;1,m)$. For the special choice of $P(s;1,m)$ discussed in Ref. [9] it was found that

$$\tilde{Q}^{(2)}_{L,1} \sim L^{\frac{5}{2}}, \quad \text{for } L \gg 1. \quad (14)$$

This is the leading order term of the second moment, $Q^{(2)}_{L,1}$, which implies $\gamma_2 \equiv D(3-\tau) = 5/2$, leading to $\tau = 4/3$ and $D = 3/2$, which characterizes the universality class of the TAOM.

We now show that Eq. (14) is true for *any* 'allowed' $P(s;1,m)$ - meaning that it was generated by toppling rules obeying the 'restrictions' (i)-(iv) and lead to a unique, non-trivial stationary state. It can be shown [13] that the random walker described by Eq. (13b) satisfies the sufficient conditions for a central limit theorem for dependent random variables [7]. For any allowed $P(s;1,m)$, there is a function, $\sigma(L) = \sigma_\infty L + \mathscr{O}(1/L)$ where σ_∞ is a non-zero constant which depends only on $P(s;1,m)$, such that $\psi_{L,k}\left[s/\sigma(L)\right]$ converges in distribution to a unique probability distribution. Hence, we find that the scaling in Eq. (14) is independent of the specific form of $P(s;1,m)$ and we recover the exponents $\tau = 4/3$ and $D = 3/2$.

The scaling of the TAOM will only be observed asymptotically for $L \to \infty$. However, we hypothesize the existence of an n dependent crossover length, ξ_n, such that TAOM scaling is observed for $L \gg \xi_n$, with different behavior for $L \ll \xi_n$. To see what happens for system sizes $L \ll \xi_n$, consider a system which has been charged once. If the probability, p_z, that a site is occupied by z particles has support for all $z \in [0, n-1]$, then, for $n \gg 1$, it is likely that $0 \ll z \ll n-1$. If the number of particles on a site z is neither close to 0 nor $n-1$, the propagating avalanches will not be sensitive to the medium within which it is propagating and the system will be temporally uncorrelated. Assume the first site, with z_1 particles, topples s_1 times when charged. s_1 will have little dependence on the number of previous topplings as they all occur in the regime $1 \ll z_1 \ll n-1$, that is, the topplings will be uncorrelated. The second site has now been charged s_1 times and with each charging has a probability of toppling a number of times, with a total of s_2 topplings. We can apply the same arguments for the dependence of s_2 on previous topplings as for s_1, except that it is now less likely that z_2 will remain in the region $1 \ll z_2 \ll n-1$ because s_2 will fluctuate more than s_1. Continuing this argument, we see that, as fluctuations increase, each subsequent site is less likely to have uncorrelated topplings and sites start to feel the fact that n is finite. Hence, for small system sizes, $1 \ll L \ll \xi_n$, the avalanches are uncorrelated and will correspond to an uncorrelated branching process with exponents $\tau = 3/2$ and $D = 2$ [10]. For large system sizes, $L \gg \xi_n$, temporal correlations will emerge and the system will crossover to behavior of the TAOM with $\tau = 4/3$ and $D = 3/2$.

We have argued that a central limit theorem leads to universality in directed sandpile models. However, a number of assumptions have been made and so it is necessary to support these claims with numerical evidence. We present data from the following 'typical' realization: A site i, $0 < z_i < n-1$, which is charged will topple once with probability $1/2$ and twice with probability $1/4$. If $z_i = 0$, then it cannot topple twice and will topple once with probability $1/2$. If $z_i = n-1$, it will topple once or twice, each with probability $1/2$. It can be shown that the support of p_z extends over all possible states $z \in [0, n-1]$. We shall compare numerical results from this model with exact results from the corresponding uncorrelated branching process, which we can calculate analytically [10].

Figure 1(a) displays measurements of the rescaled second moment, $Q^{(2)}_{L,1}/L^{5/2}$ vs. L. From the arguments above, we expect $Q^{(2)}_{L,1}/L^{5/2}$ to scale like $L^{1/2}$ for $1 \ll L \ll \xi_n$ and approach a constant for $L \gg \xi_n$, which is supported by the numerics. We do not attempt a data collapse because, although it is clear that the crossover length, ξ_n, is a non-decreasing function of n, it has a non-universal functional dependence.

We also consider the moment ratios

$$g_k \equiv \frac{\langle s^k \rangle \langle s \rangle^{k-2}}{\langle s^2 \rangle^{k-1}} \equiv \frac{\Gamma_k \Gamma_1^{k-2}}{\Gamma_2^{k-1}}. \quad (15)$$

For an avalanche-size probability of the form

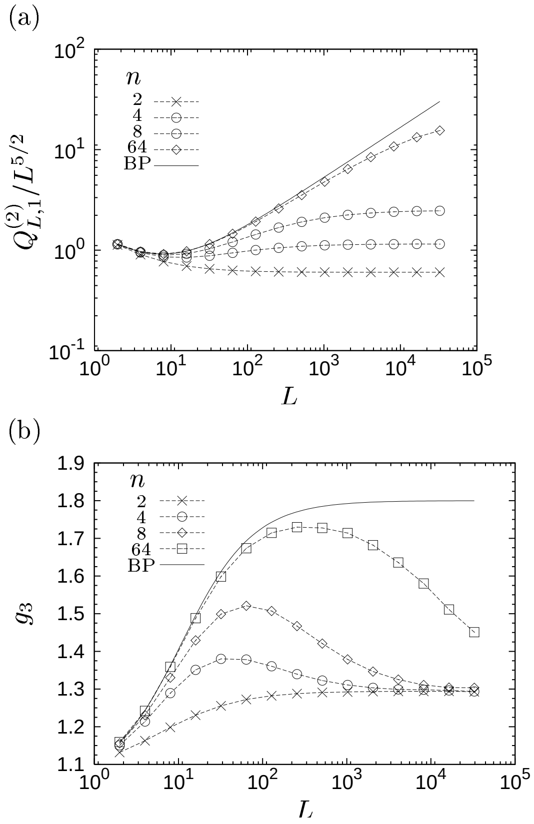

FIG. 1: Numerical results for a typical realisation with $n = 2, 4, 8, 64$ and exact results for the uncorrelated branching process (BP). The errors for both graphs were calculated using Efron's Jackknife [11] and are smaller than the symbols.
(a) The rescaled second moment, $Q^{(2)}_{L,1}/L^{5/2}$, vs. system size, L. For $n = 2, 4, 8, 64$, the rescaled second moment approached a constant for $L \gg 1$, while for the branching process it increases like $L^{1/2}$.
(b) The moment ratio, g_3, vs. system size, L. For $n = 2, 4, 8, 64$, the moment ratios follow the branching process curve for $L \ll \xi_n$ before approaching the TAOM value of $g_3 \approx 1.29$ for $L \gg \xi_n$.

Eq. (1), g_k will be universal constants, that is, they only depend on τ and $\mathscr{G}$. Figure 1(b) displays g_3 vs. system size, L. Since $G_k(L)$ is only constant for $L \gg 1$, the measured g_3 will only converge toward the universal constant for large L. These have values $g_3 = 9/5$ for the branching process and we measure $g_3 \approx 1.29$ for the TAOM. For $1 \ll L \ll \xi_n$, the measured values follow the exact result for the uncorrelated branching process, with a crossover to the TAOM curve for $L \gg \xi_n$.

Finally, we have checked the validity of the assumptions made throughout this work on various realizations of the model and find that they are all very well justified.

We have argued and shown numerically that a general n-state sandpile model of self-organized criticality belongs to the universality class of the totally asymmetric Oslo model, recently solved in Ref. [9]. We have argued that the number of states, n, is a relevant parameter such that there is an n dependent crossover length, ξ_n, which separates uncorrelated branching process exponents, $\tau = 3/2$ and $D = 2$, for $1 \ll L \ll \xi_n$ from TAOM exponents, $\tau = 4/3$ and $D = 3/2$ for $L \gg \xi_n$. This crossover may be considered a consequence of temporal correlations emerging in the system, which moves it away from the uncorrelated branching process, associated with mean-field exponents. The origin of the universality was shown to be a central limit theorem for dependent random variables [7]. This is the first time a technique of applying a central limit theorem has been used to identify a universality class of non-equilibrium self-organized critical systems and we expect much new research will follow along these lines.

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