Exact enumeration of the Critical States in the Oslo Model

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We determine analytically the number $N_{\mathcal{R}}(L)$ of recurrent states in the $1d$ Oslo model as a function of system size L. The solution $N_{\mathcal{R}}(L) = \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^L + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^L$ is in exact agreement with the number enumerated in computer simulations for $L = 1-10$. For $L \gg 1$, the number of allowed metastable states in the attractor increases exponentially as $N_{\mathcal{R}}(L) \approx c_+\lambda_+^L$, where $\lambda_+ = \frac{3+\sqrt{5}}{2}$ is the golden mean. The system is non-ergodic in the sense that the states in the attractor are not equally probable.

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Some non-equilibrium systems can be described by simple physical laws. Despite this being the case, we find complex and non-linear behavior in many such systems. A slowly driven granular system illustrates this phenomenon spectacularly. If grains are deposited onto a finite surface, they will eventually build into a pile. The pile does not collapse and flatten due to the friction between the grains. We call the different profiles of piles produced by this process, metastable states. The addition of grains and the dissipation events (avalanches) form a delicate feedback loop. Avalanches initiated by dropping grains onto the pile will ensure that the slope does not become too steep, while the addition of grains prevents the slope from becoming too shallow. Eventually, the system settles into an attractor where, the average avalanche size of the system diverges in response to small perturbations induced by adding grains. The distribution of avalanche sizes becomes scale invariant and a small perturbation may initiate small as well as large avalanches. As such, the system is intrinsically non-linear and displays characteristics similar to equilibrium systems poised at a critical point. This has been observed experimentally, for real granular systems where friction dominates over inertial effects [1], as well as numerically [2, 3]. It is an example of systems displaying self-organized criticality [2, 4, 5].

We study the attractor of a simple $1d$ granular model known as the Oslo model. It successfully models slowly driven granular systems displaying self-organized criticality [1, 3]. The model has been shown to be a member of a large universality class of $1d$ systems, which includes the de-pinning transition of an interface dragged through a random medium and the deterministic Burridge-Knopoff train model for earthquakes [6–8]. Thus, the Oslo model plays the role of the "Ising model" for $1d$ self-organized critical systems and has been widely studied [6, 8–20]. However, in contrast to various deterministic models of granular systems where exact results do exist [21, 22], the Oslo model has resisted analytic treament [9]. Only a handful of exact results are known for non-equilibrium systems in general [23] and self-organized critical systems in particular [21, 22, 24]. Any analytic result for the Oslo model would be very valuable.

We are able to determine the number $N_{\mathcal{R}}(L)$ of states in the attractor of the Oslo Model. We show that $N_{\mathcal{R}}(L) = \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^L + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^L$. For system sizes $L \gg 1$, the number increases exponentially fast as $N_{\mathcal{R}}(L) \approx c_+\lambda_+^L$, where λ_+ is the golden mean. This is a clear signature of the complexity displayed by this simple model. Moreover, the system is non-ergodic. The recurrent states are not visited with the same frequency.

The Oslo model is defined on a one-dimensional discrete lattice consisting of L sites, $i = 1, 2, \ldots, L$. There is a vertical wall at the left boundary next to site $i = 1$. The system is open at the right boundary next to site $i = L$, where grains are allowed to leave the system. We will denote the respective heights of the pile starting from the vertical wall by an n-tuple $\mathbf{h} = (h_1, h_2, \ldots, h_i, \ldots, h_L)$, see Fig. 1. In accordance with the experiment in Ref. [1],

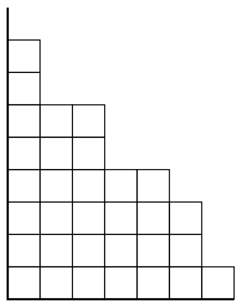

FIG. 1: The Oslo model of a $1d$ granular pile. An integer height $h_i \geq 0$ is assigned to each site i. Grains are added at the site $i = 1$ next to the vertical wall by letting $h_1 \to h_1 + 1$. Grains can leave the system at the open boundary, $h_L \to h_L - 1$.

grains are only inserted at $i = 1$, the site next to the vertical wall. In the following discussion, it will be convenient to refer to local slopes instead of heights. The slopes are denoted by $\mathbf{z} = (z_1, z_2, \ldots, z_i, \ldots, z_L)$ with the definition $z_i = h_i - h_{i+1}$ where $h_{L+1} = 0$. Each site i is assigned a critical slope, $\mathbf{z}^c(t) = (z_1^c(t), z_2^c(t), \ldots, z_i^c(t), \ldots, z_L^c(t))$. If the local slope exceeds the critical slope, $z_i > z_i^c(t)$, a grain at site i will topple to the site $i+1$, that is

$h_i \to h_i - 1, h_{i+1} \to h_{i+1} + 1$. As a consequence, the slopes at neighboring sites $i \pm 1$ may also exceed their critical slopes and topple, consequently causing an avalanche that propagates until the system settles in a metastable state with $z_i \leq z_i^c$ for all i.

The critical slope $z_i^c(t)$ is a function of time as it is chosen randomly every time site i topples. We denote the minimum allowed value of the random critical slopes $z^{c(min)}$ and similarly the maximum allowed value $z^{c(max)}$. In general, we can use any discrete probability distribution of critical slopes. However, we will consider the simplest case where z_i^c is chosen at random between the 1 or 2, $z_i^c(t) \in \{1, 2\}$ when site i topples, such that $z^{c(min)} = 1$ and $z^{c(max)} = 2$.

The algorithm for the Oslo model is defined as follows.

1. Initialize the critical slopes z_i^c and the system in a arbitrary metastable state with $z_i \leq z_i^c$ for all i.

2. Add a grain at site $i = 1$, i.e., $z_1 \to z_1 + 1$.

3. If $z_i > z_i^c$, the site relaxes and
$$z_i \to z_i - 2$$
$$z_{i\pm1} \to z_{i\pm1} + 1$$
except for the sites at the vertical wall $i = 1$ and the open boundary $i = L$ where
$$z_1 \to z_1 - 2 \qquad\qquad z_L \to z_L - 1$$
$$z_2 \to z_2 + 1 \qquad\qquad z_{L-1} \to z_{L-1} + 1.$$
The critical slope at site i is chosen randomly if it topples, that is $z_i^c \to 1$ or 2. A new metastable state is reached when $z_i \leq z_i^c$ for all i.

4. Proceed to 2 and reiterate.

Note the separation of time scales which is built into the definition of the model. The addition of grains can only take place when the system has settled down into a metastable state. Thus the response of the system (the avalanches) is fast when compared with the interval between perturbations.

Metastable states $\mathcal{M}_j$ are rest states of the system. Different metastable states occur due to the different possible sets of critical slopes. Metastable states can also be divided into transient states $\mathcal{T}_j$ and attractor states $\mathcal{R}_j$. Transient states are not reachable once the systems has entered into the attractor, which are recurrent metastable states. The index j denotes discrete time steps associated with the long time scale of the system.

When adding a grain to a metastable state $\mathcal{M}_j$, it evolves into a new metastable state $\mathcal{M}_{j+1}$ by the relaxation rules given above. Symbolically we write $\mathcal{M}_j \mapsto \mathcal{M}_{j+1}$, where the arrow is a shorthand notation for the operation of adding sand and, if necessary, relaxing the system until it reaches a new metastable state. Starting from, say the empty lattice $\mathcal{T}_1$, which is a transient state as it will never be encountered again, we have

$$\mathcal{T}_1 \mapsto \cdots \mapsto \mathcal{T}_n \mapsto \mathcal{R}_1 \mapsto \cdots \mapsto \mathcal{R}_{j-1} \mapsto \mathcal{R}_j \mapsto \mathcal{R}_{j+1} \mapsto \cdots$$

After n additions and associated relaxations (if any), the system reaches the attractor of the dynamics. For example, if we set z_i^c independent of time, we arrive at the BTW model in one dimension [2]. Any grain introduced into the system after the attractor state has been reached will simply flow to the open boundary fall out. In this case, there is only one attractor state with $z_i = z_i^c \ \forall i$. We want to enumerate the number of different states in the attractor $\{\mathcal{R}_j\}$ for the Oslo model, where z_i^c is chosen randomly between 1 or 2.

We adopt the following nomenclature for sites

$$\begin{array}{rll} z_i < z^{c(min)} & \text{sinks}, & z_i = z^{c(min)} \ \text{stable}, \\ z^{c(min)} < z_i \leq z^{c(max)} & \text{critical}, & z_i > z^{c(max)} \ \text{supercritical}. \end{array}$$

Due to the way the problem is defined, $z_i \geq 0$ always. Also, supercritical sites are by definition not allowed in any metastable state. Fig. 2 shows a diagrammatic relation of these classifications. From the relaxation rules,

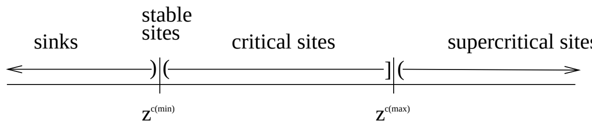

FIG. 2: We classify different types of sites according to the value of the local slope $\mathbf{z}_i$. The above diagram shows these definitions in relation to one another, where $z^{c(min)}$ and $z^{c(max)}$ are the minimum and maximum allowed values for critical slopes respectively.

we find that states in the attractor are subject to the following two constraints.

(C1) Starting from the open boundary $i = L$, the first site ($i < L$) that is not a stable site ($z_i \neq z^{c(min)}$) must be a critical site.

(C2) The first site that is not a stable state to the left of the sink must be a critical site. Equivalently, we cannot find a recurrent state in which two sinks are separated only by stable sites. The exception is at the closed boundary $i = 1$, where the first site that is not stable must be the boundary itself.

We can show that these constraints must be fulfilled by assuming that each is not true and examining how that system would enter such as state. For example, Fig. 3(a) shows a case where the the first rule **C1** does not hold. The possible preceding configurations are displayed in Fig. 3(b) and (c). Since these are not states in the attractor, the state displayed in Fig. 3(a) is not in the attractor. One could argue likewise for the second constraint, **C2**. Note however, that by iterating these cases, we have only shown that these constraints are necessary but not sufficient. However, we make the conjecture that these constraints are also sufficient for enumerating all attractor states from exact agreement with simulations for system sizes $L = 1$ to 10, see Fig. 4.

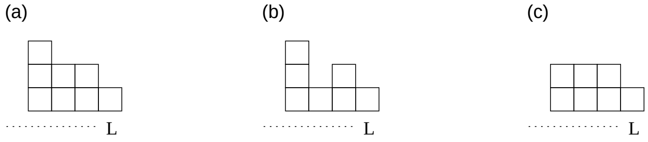

FIG. 3: (a) The configuration displayed violates the constraint (**C1**), as the first site that is not stable (here, $i = L-2$) next to the open boundary $i = L$ is a sink, $z_{L-2} = 0 < z^{c(min)}$. (b) and (c) are two possible preceding configurations. These configurations do not belong to the attractor.

For a system with L lattice sites, we want to find the number of states $N_{\mathcal{R}}(L)$ that obey the constraints above. We can simplify the problem further by first considering the number states with a fixed height at site $i = 1$, h_1. We introduce the parameter $q = h_1 - L = \sum_{i=1}^{L} z_i - L$ as it simplifies the equations involved considerably. Observe that the range of q is $0 \leq q \leq L$. Let $K(L, q)$ describe the number of states in a system of L sites and $h_1 = q + L$. $N_{\mathcal{R}}(L)$ can then be determined by $N_{\mathcal{R}}(L) = \sum_{q=0}^{L} K(L, q)$.

To generate all states $\{\mathbf{z}\}$ that obey the counting constraints for states in the attractor, consider the situation where we know all the allowed profiles for system sizes $L' < L$. We denote the family of n-tuples that represent allowed profiles for a system of size L by $\mathcal{F}(L)$. $N_{\mathcal{R}}(L)$ is the number of n-tuples in $\mathcal{F}(L)$. We can generate $\mathcal{F}(L) = \{\mathbf{h}\}$ by extending $\mathcal{F}(L-1)$ by one site at the closed boundary. This may be done using the following procedure - extend each $\mathbf{z}'$ to $\mathbf{z}$ by writing $z_{i+1} = z'_i$. Consider one of three possible values for the new site next to the closed boundary, z_1. $z_1 = 2$ for critical sites, $z_1 = 1$ for stable sites and $z_1 = 0$ for sinks. Lastly, check that the new states in each of the three cases obey the counting constraints (**C1**) and (**C2**).

Consider the three possible values for z_1, that give $\mathcal{F}(L)$ by extending $\mathcal{F}(L-1)$ corresponding to three possible cases such that $K(L, q) = \sum_{i=1}^{3} K^{(i)}(L, q)$.

1. When $z_1 = 2$, there is a critical site next to the closed boundary. The number of allowed profiles for a system of size L and parameter q must be the same as the number of allowed profiles with a system size of $L' = L - 1$. We relate q' to q by observing that $q = h_1 - L = h'_1 - (L-1) + 1 = q' + 1$. Therefore we assert that when $z_1 = 2$, $K^{(1)}(L, q) = K(L-1, q-1)$.

2. Similarly, when $z_1 = 1$ corresponding to a stable site next to the closed boundary all profiles for a system size $L - 1$ with $q' = q$ are allowed. In this case, $K^{(2)}(L, q) = K(L-1, q)$.

3. Finally, the situation when $z_1 = 0$ means that a sink resides next to the closed boundary. The counting constraints state that the sink next to the closed boundary must be followed by a critical site with any number of stable sites in between. In this case, we consider $\mathbf{z}' = (0, 2, \dots)$ extending $\mathcal{F}(L-2)$, $\mathbf{z}' = (0, 1, 2, \dots)$ extending $\mathcal{F}(L-3)$ and so on to give L. This involves summing all allowed metastable states $\mathcal{F}(L')$ with $L' \leq L-2$. As above, we find that $q' = q$ for all L'. Hence
$$K^{(3)}(L, q) = \sum_{i=1}^{L-2} K(i, q).$$
Note that this equation is a simple summation because of the appropriate choice of q.

Summing the three cases,

$$\begin{aligned} K(L,q) &= K(L-1,q-1) + K(L-1,q) + \sum_{i=1}^{L-2} K(i,q) \\ &= \sum_{i=1}^{L-1} K(i,q) + K(L-1,q-1) \\ &= 2K(L-1,q) + K(L-1,q-1) - K(L-2,q-1). \end{aligned}$$

Hence we calculate the number of states as a function of system size by summing through q

$$\begin{aligned} N_{\mathcal{R}}(L) &= \sum_{q=0}^{L} K(L,q) \\ &= 3\sum_{q=0}^{L-1} K(L-1,q) - \sum_{q=0}^{L-2} K(L-2,q) \\ &= 3N_{\mathcal{R}}(L-1) - N_{\mathcal{R}}(L-2). \end{aligned}$$

We have used the fact that $0 \leq q \leq L$. $N_{\mathcal{R}}(L) = 3N_{\mathcal{R}}(L-1) - N_{\mathcal{R}}(L-2)$ is a Fibonacci like relation. Substituting a trial solution $N_{\mathcal{R}}(L) = c_+\lambda_+^L + c_-\lambda_-^L$, and demanding the initial conditions $N_{\mathcal{R}}(1) = 2$ and $N_{\mathcal{R}}(2) = 5$, we obtain the solution, see Fig 4.

$$N_{\mathcal{R}}(L) = \frac{1+\sqrt{5}}{2\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^L + \frac{\sqrt{5}-1}{2\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^L. \tag{1}$$

The golden mean $\lambda_+ = \frac{3+\sqrt{5}}{2} \approx 2.618\dots$. Computer simulations were used to test these results for systems of size 1 to 10. They were found to be in full agreement.

This result should be compared with the size of the attractor for the abelian Bak-Tang-Wiesenfeld sandpile determined by Dhar [21]. He showed that $N_{\mathcal{R}} = \mathrm{Det}\Delta$, where Δ is the toppling matrix. In $2D$ and for large system sizes, $N_{\mathcal{R}}$ increases exponentially fast with the number of sites in the system $(3.21\dots)^{L\times L}$ [25]. Moreover, it was proven that the BTW model is ergodic [21]. We find, in contrast, that the Oslo model is non-ergodic in the sense that the states in the attractor are not visited with the same frequency. This is most easily demonstrated by showing that the probability of visiting stationary states in very small systems (e.g., $L = 2$ which has a total

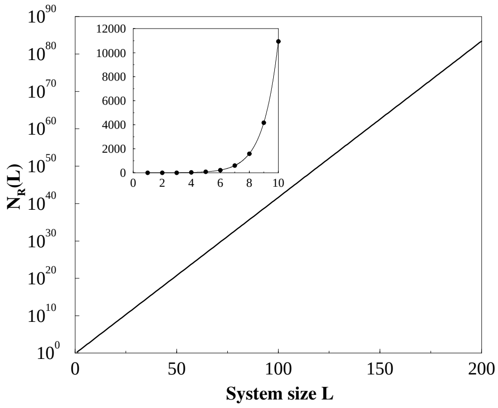

FIG. 4: Inset: Both simulation and analytic results are shown for systems with 1 to 10 lattice sites. Results obtained by enumerating the different metastable states in the attractor (filled circles) are identical to analytic results in Eq. (1) (solid line). Main figure: The analytic solution of Eq. (1) showing the number of states in the attractor as a function of system size. For a moderate system size of $L = 195$, the number of metastable states in the attractor exceeds 10^{81}, the estimated number of atoms in the universe.

of 5 states in the attractor) is non-uniform. The number of states in the attractor $N_{\mathcal{R}}$ does not depend on the probability distribution with which z_c^i is chosen to be 1 or 2, except for the trivial cases of $P(z_c^i = 1) = 0$ or 1, but the distribution will affect the accessibility of a given microstate. Finally, note that the Oslo model can be mapped into an interface dragged through a random medium [6]. Thus, the results obtained for the Oslo model are directly applicable to the latter model as well.

Possible extensions. A pile can be divided into an inactive zone consisting of grains that never will take part in the dynamics and the active zone. The width λ_L of the active zone plays an important role as it is related to various physical phenomena. E.g. the average transit times of individual grains has been measured to increase with system size as $\langle T \rangle_L \propto L\lambda_L \propto L^{1+\chi}$. For the Oslo model $\chi \approx 0.25$ while experimentally $\chi \approx 0.5$ [1, 3]. We believe that using a similar approach one might be able to calculate analytically the distribution of the height of the pile at the vertical wall $P(h_1)$ and the critical exponent χ. This would be of great importance as this exponent will be the same in all the models belonging to the universality class of the Oslo Model.

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