

On the avalanche size distribution in the BTW model

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Abstract

In the analysis of the probability distribution $P(s, L)$ of avalanche sizes in the two-dimensional Bak-Tang-Wiesenfeld (BTW) model, two distinct regions can be resolved for large system sizes L . The region at large avalanche sizes s has the characteristics of a finite-size cut-off similar to that observed more clearly in other models and is thus excluded from the scaling analysis. At computationally available system sizes, the critical exponents $\tau_{s,L}$ of the remaining regions display a proportionality to $\log L$ rather than the previously reported $1/\log L$ and no limiting value $\tau_{s,\infty}$ for infinite system size can be determined. Further computation will be required to obtain the complete form and any asymptotic value of the scaling of τ_s and to subsequently reassess universality in this class of systems.

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The concept of Self-Organized Criticality (SOC) introduced in 1987 by Bak, Tang, and Wiesenfeld (BTW) describes a class of systems in which the critical state is an attractor of the dynamics [1,2]. In this critical state, system properties should be described by power-law or other scale-invariant distributions. Attempts have been made to assign these systems to universality classes, where models in a class are described by the same set of critical exponents. A universality class has been established for variants of the Oslo model [3,4] and a proposed class containing the BTW, Manna [5], and Zhang [6] models has been discussed from both theoretical and numerical results with conflicting conclusions [7–14]. The existence of such universality classes would point to a common underlying mechanism in a wide range of systems and make models of SOC a valuable tool in analyzing scale-invariant systems.

The two-dimensional BTW model discussed here is defined as an array of sites with integer heights z_i , $i = 1, \dots, L^2$ on a square lattice. Single grains are added to random sites i such that $z_i \rightarrow z_i + 1$. A site reaching height $z_i > z_c$ is unstable and topples, distributing one grain to each of its four nearest neighbors, $z_{nn} \rightarrow z_{nn} + 1$, and assuming height $z_i \rightarrow z_i - 4$. These rules allow avalanches of topplings to propagate through the system. The BTW model has been examined extensively both numerically and analytically with a focus on the probability distribution of avalanche sizes, where the avalanche size s is defined as the total number of topplings during an avalanche [15,16]. The corresponding probability distribution $P(s, L)$ for a system with a given linear extent L is expected from theory to follow a power law with exponent $\tau_{s,L}$, although it has recently been suggested from numerical results that $P(s, L)$ is multifractal such that no unique $\tau_{s,L}$ can be defined [17,18].

In this letter, we reexamine the form of $P(s, L)$ and the scaling of the resulting $\tau_{s,L}$ with L and demonstrate a deviation from all previously reported scaling behaviour. The finite-size scaling assumption in the form proposed by Manna, $\tau_{s,L} = \tau_{s,\infty} - \frac{\text{constant}}{\log_{10} L}$, is based on an empirical argument and would allow $\tau_{s,\infty}$ to be found by linear extrapolation [19]. More recent results give evidence of a violation of FSS in the computationally available region, noting particularly the influence of boundary effects [17,18,20]. We show that for

sufficiently good statistics, the region of $P(s, L)$ in which boundary effects dominate can be clearly identified and discounted. The remaining values are fully consistent with the existence of a $\tau_{s,L}$ at every L and under this assumption, good estimates can be extracted.

In any numerical simulation of the BTW model, data at large s will suffer from high statistical noise, even for data sets with very large numbers of avalanches. To reduce this noise the data is assigned to bins of exponentially increasing size along the s -axis, with the exponential increase ensuring an approximately equal number of data points in each bin. The values of the local slope, i.e. $|d(\log_{10} P(s, L))/d(\log_{10} s)|$, are obtained from the binned values (see Fig. 1) and a histogram indicating the frequency of their occurrence is created to give an objective measure of the best estimate of $\tau_{s,L}$ and the associated uncertainty (see Fig. 2). Pure power-law behaviour would result in a narrow spike centered on the value of $\tau_{s,L}$. Instead, the peak broadens with increasing L and can eventually be resolved as a double peak for $L = 2048$ in runs of 10^7 avalanches. Inspection of plots of the local slope against s confirm that the peaks correspond to two distinct power-law regions. The flatter region at larger s can be thought of as part of a cut-off, similar to that observed in other SOC models such as the Oslo model [21,22]. In this cut-off regime, avalanche extent is truncated by the boundaries and the dynamics are dominated by multiple topplings. This cut-off interpretation is supported by the observation that the onset of the flatter region moves to larger s approximately with L^2 . The two peaks are less clearly resolved for system sizes below $L = 2048$ and could be interpreted as a single broad peak. This interpretation would render the central value of the peaking region the best estimate of $\tau_{s,L}$ and Fig. 3 shows that this interpretation results in only small deviations from the FSS scaling assumption. These central values of $\tau_{s,L}$ represent the mean value around which $|d(\log_{10} P(s, L))/d(\log_{10} s)|$ fluctuates in the scaling region in Fig. 1. Extrapolation of these values to an infinite size lattice gives $\tau_{s,\infty} \approx 1.22$, in agreement with other published results that assume FSS [19,23].

For a revised analysis taking account of the two distinct regions, values of the higher and the lower sections of $|d(\log_{10} P(s, L))/d(\log_{10} s)|$ can be determined from Fig. 1 for all system sizes. For $L = 2048$ these differ considerably and clearly correspond to the two peaks

in Fig. 2, whereas for small systems the deviation from the central value is undetectable. The observed higher regions at all L are consistent with power-law behavior and thus a value for $\tau_{s,L}$ can be extracted.

With the above interpretation of the lower value as a cut-off, the scaling is reexamined using only the higher values at each system size as the best estimate for $\tau_{s,L}$. These values show a clear deviation from FSS (see Fig. 3) and plotting them against $\log_{10} L$ indicates a proportionality to $\log_{10} L$ in the region of available results (see Fig. 4).

The proportionality to $\log_{10} L$ observed for $\tau_{s,L}$ with the above analysis raises fundamental questions in attempts to interpret it. To achieve consistency with the existence of a limiting value $\tau_{s,\infty}$ for infinite system size, the scaling would have to include a deviation at larger L from the observed behaviour. An indefinite increase of $\tau_{s,L}$ cannot be reconciled with SOC, as for $\tau_{s,L} > 2$, $P(s, L)$ would acquire a finite first moment for infinite system size, in contradiction to analytical results [24]. A restriction of the proportionality would therefore be expected. To confirm any such restriction and attribute the observed scaling to finite-size effects only, further data at larger L will be essential. The present results do not show evidence of a limiting value of $\tau_{s,L}$ for infinite system size and hence do not allow any classification of the universality of the model based on numerical results.

No specific explanation for the observed $\log L$ dependency can be offered, but contributing finite-size effects can be identified at both small and large s . At large s , the deviation is formed by the obvious cut-off in avalanche size together with the adjacent flatter scaling region. For $L = 2048$, a bend in $P(s, L)$ is clearly observable where the flatter cut-off behaviour starts to dominate. This can also be seen directly in plots of $P(s, L)$, e.g. $P(s, 4096)$ in Ref. [23]. For the Zhang model, a similar bend in $P(s, L)$ has been reported [25]. At small s , deviations from power-law scaling are caused by the scale-dependent dynamics of the model. The restricted number of combinations of elementary processes in small systems leads to significant deviations from the expected asymptotic scale-invariance. At the smallest scale, the elementary cell has fixed toppling behaviour, always throwing grains to all the nearest neighbor sites. The transition from this towards the rule set at infinite scale can be

demonstrated by considering the minimum number of sides that are involved in a toppling process at each of the first four scales using the dynamically driven renormalization group (DDRG) formalism introduced in Ref. [12,13].

In the full DDRG calculation, the first iterations, corresponding to the scales shown in Fig. 5, correctly reproduce this behaviour. Both numerical results and DDRG calculations therefore suggest that this scale-dependence is a significant element of the scaling behaviour for all computable system sizes, rather than a perturbation restricted to very small s .

Due to the cut-off effects at both small and large s , the region in which $P(s, L)$ displays its true scaling behaviour is smaller than a first inspection would suggest. In particular at small L , the effect of scale-dependent dynamics is still significant up to the large s cut-off region. With increasing L , a region where neither effect is significant is expected to be recovered. Observations of any approach of $\tau_{s,L}$ to $\tau_{s,\infty}$ will be influenced by this, as the measured value for $\tau_{s,L}$ only gradually approaches the true exponent corresponding to the respective system size. This gradual approach cannot be expected to follow FSS and hence FSS is not expected to be observable at the available system sizes, irrespective of its general validity for the model.

In conclusion, our results underline the influence of finite-size effects at both small and large scales on the obtainable results for $\tau_{s,L}$. It cannot be concluded whether these effects are the cause of the observed scaling behaviour or just partly obscure more fundamental mechanisms. As a result, significant further computation will be required to understand the scaling behaviour of the two-dimensional BTW model and thus obtain robust results on the existence and any possible value of $\tau_{s,\infty}$. Such a result would be a necessary base for the discussion of universality in this class of systems.

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FIGURES

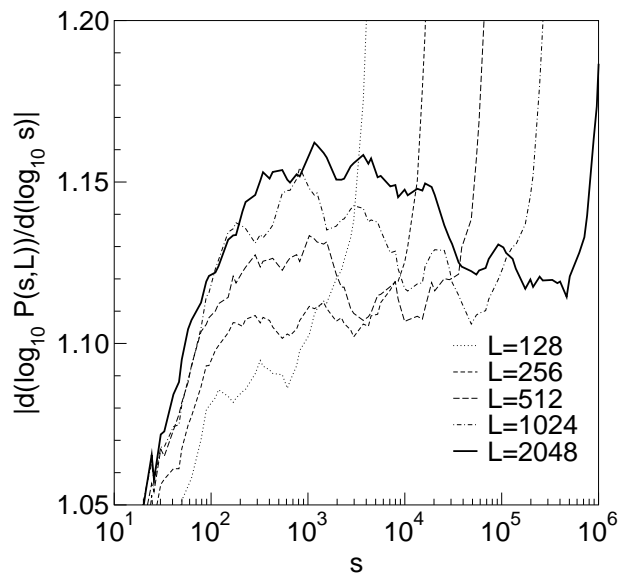


FIG. 1. Local slope for the 2D BTW model for system sizes $L = 128, 256, 512, 1024,$ and 2048 . For $L = 2048$, the two distinct regions corresponding to the peaks in Fig. 2 are clearly observed. For smaller systems, the higher value is taken as the local average of the scaling region at lower s .

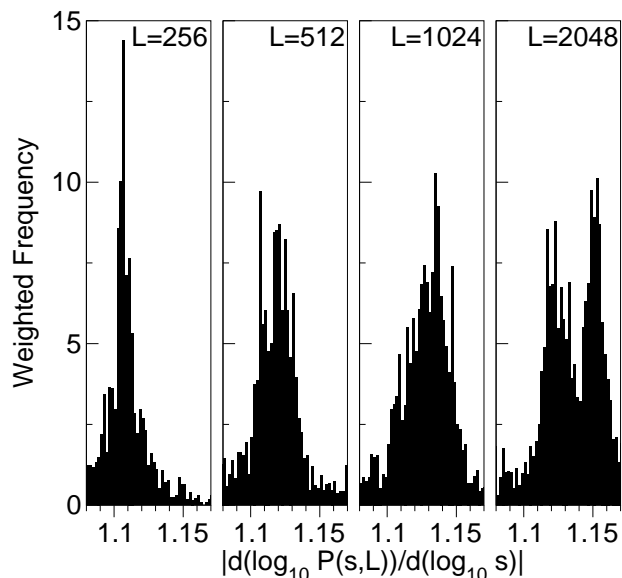


FIG. 2. Histograms of binned slope values for the 2D BTW model with $L = 256, 512, 1024,$ and 2048 , with 10^7 avalanches calculated at each L . Only $L = 2048$ shows two distinct peaks. The histograms use data points calculated from a set of binnings with different bases and offsets, with the data points weighted such that each base size makes the same overall contribution.

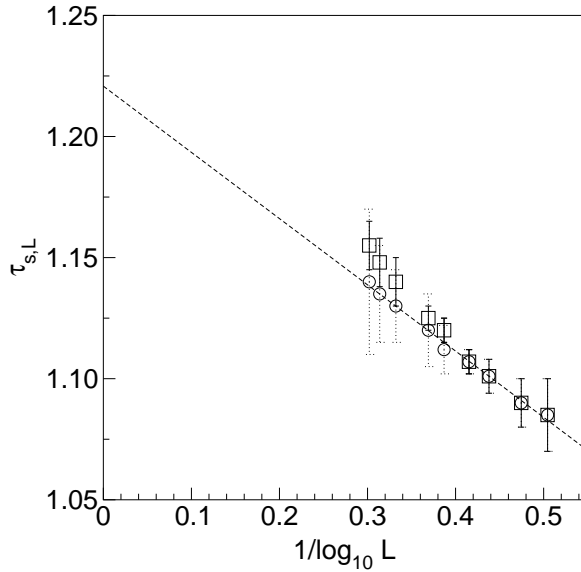


FIG. 3. Scaling of $\tau_{s,L}$ with $1/\log_{10} L$ for the 2D BTW model using central values (\circ) and higher values (\square) for $L = 96$ to $L = 2048$ and 10^7 avalanches at each L . Also shown is a least squares fit to the central values, illustrating the good agreement of the central values with the assumption of FSS.

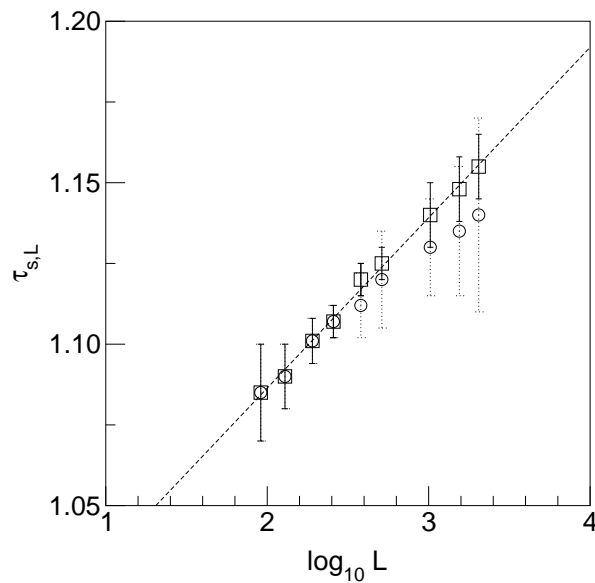


FIG. 4. Data of Fig. 3 plotted as $\tau_{s,L}$ against $\log_{10} L$. The least squares fit to the higher values (\square) shows a proportionality to $\log L$.

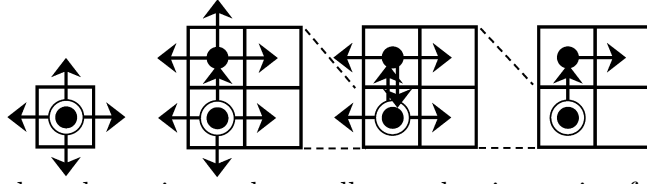


FIG. 5. Scale-dependent dynamics at the smallest scales, increasing from left to right. Shown are the minimally dissipative process series at $k = 0, 1, 2, 3$, where at each scale, the linear extent compound cell is equal to 2^k elementary cells. Dots indicate critical, circled dots supercritical sites, in accordance with the formalism in [12,13]. It can be seen that $k = 3$ is the smallest scale at which the possible combinations of elementary processes allow a purely internal spanning toppling. This scale-dependence of the dynamics continues at larger scales and has significant effect on the observable scaling.