DYNAMICAL ASPECTS OF SANDPILE CELLULAR AUTOMATA

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ABSTRACT. The Bak, Tang, and Wiesenfeld cellular automaton [Phys. Rev. Lett. **59**, 381, (1987)] is analysed with respect to the dynamical behavior. The scaling of the power spectrum derives from the assumption that the instantaneous dissipation rate of the individual avalanches obeys a simple scaling relation. Primarily, the results of our work show that the flow of sand down the slope does not have a $1/f$ power spectrum in any dimension. The power spectrum behaves as $1/f^2$ in all the dimensions considered.

1. Introduction

It has been a long-standing puzzle why $1/f$ power spectra are seen in a variety of physical systems. Also, the occurrence of spatial fractal structures has been realized as an empirical fact in many different systems, although a proper understanding of the physical origin is still lacking. In a recent paper by Bak, Tang, and Wiesenfeld [1] (BTW), it was suggested that the frequent occurrence of $1/f$ noise and fractal structures is the generic temporal and spatial characteristic of a dynamical critical state into which dynamical systems with many spatial degrees of freedom evolve naturally. Unlike phase transitions in a equilibrium system, a driven dissipative dynamical many-body-system reaches the critical state without the need to fine tune the system parameters, i.e., the critical state studied by BTW is an attractor of the dynamics. This phenomena of self-organized criticality (SOC) may very well provide a connection between the occurrence of $1/f$ noise and fractal structures, as well as being the physical origin of these two intriguing phenomena.

In order to visualize the basic idea of self-organized criticality: Imagine a sunny day at the beach and a square table. We begin sprinkling grains of sand on randomly chosen places on the table, one grain at a time. Eventually, we end up with only one big sandpile. At some point this pile ceases to grow. The pile has reached a statistically stationary state, and additional grains of sand will ultimately fall off the pile by means of avalanches.

In order to examine the phenomenon of self-organized criticality, BTW [1] introduced a cellular automaton (CA). A CA involves discrete space coordinates and discrete time steps. Furthermore, the physical quantities, that are connected with the lattice sites, only take on a finite set of discrete values. The state of the CA is completely specified by the

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values of the physical variables on each site. The dynamical rules for the physical variables determine the evolution of the model. The dynamical rules in the BTW model, at least intuitively, resemble the dynamics of a sandpile: A signal is transmitted from a local site to its nearest neighbors when a dynamical integer variable exceeds a threshold value.

2. The Power Spectrum

Given a set of canonical basis vectors $\{\mathbf{e}_i\}, i = 1, \cdots, d$, let r_i (an integer restricted to the interval between 0 and N) denote the i-th coordinate of a point $\mathbf{r}$. To each lattice site $\mathbf{r}$ we assign an integer $z(\mathbf{r})$ which is to represent a discrete version of an appropriate dynamical variabel (ex. slope, stress, energy) on site $\mathbf{r}$ of a spatially extended dynamical system.

A point in phase space of the d-dimensional dynamical system is completely specified by the total set of dynamical variables $\{z(\mathbf{r})\}$, i.e., a trajectory in phase space corresponds to a particular evolution of the dynamical system.

The dynamical rules of the d-dimensional model is defined as folows: We consider the *non-conservative perturbation mechanism*

$$z(\mathbf{r}) \to z(\mathbf{r}) + 1 \tag{1}$$

and whenever the local slope exceeds a certain *critical slope* z_c sand tumbles. The corresponding changes in the z-values will be given according to the *relaxation* algorithm:

$$\text{If } z(\mathbf{r}) > z_c \text{ then} \qquad \begin{aligned} z(\mathbf{r}) &\to z(\mathbf{r}) - 2d \\ z(\mathbf{r} \pm \mathbf{e}_i) &\to z(\mathbf{r} \pm \mathbf{e}_i) + 1 \qquad \text{for } i = 1, \cdots, d. \end{aligned} \tag{2}$$

If several sites $\mathbf{r}$ are unstable, $z(\mathbf{r}) > z_c$, the relaxations will take place simultaneously. We impose *closed boundary conditions*, i.e.,

$$z(\mathbf{r}) = 0 \qquad \text{if there exists } r_j = 0 \text{ or } r_j = N \tag{3}$$

The algorithm of the temporal evolution of the sandpile cellular automaton follows:

1. Specify an initial configuration $\{z(\mathbf{r})\}$.
2. If any $z(\mathbf{r}) > z_c$ then
   relax the configuration simultaneously by use of Eq. (2)
   taking into account the closed boundary conditions
   until $z(\mathbf{r}) \leq z_c$ for all sites $\mathbf{r}$.
3. Choose a position $\mathbf{r}$ at random.
   Perturb the system and Return to step 2.

Since we want to analyse the response of the sandpile automaton (the flow of sand down the slope of the sandpile) due to white-noise perturbations (adding sand randomly in space and time) in the frequency domain we have to define the time concept and we define a *unit time-step* as one update of the whole latice [2].

First we concentrate on a member labelled by i of an ensemble. Introduce an indicator function of unstable sites at time τ on the lattice

$$f_i(\mathbf{r},\tau) = \begin{cases} 1, & \text{if } z(\mathbf{r}) > z_c; \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

We define the ***instantaneous dissipation rate*** of an avalanche α in the i-th member by

$$f_{\alpha_i}(\tau) = \sum_{\mathbf{r}} f_i(\mathbf{r},\tau), \tag{5}$$

In other words, we assign to each avalanche (outcome) a function of time, whose value at time τ equals the total number of relaxations at that instant.

We consider a system in the stationary state and perturb it by adding sand, on randomly chosen lattice sites, with a constant probability ν per time. For a sufficiently large system we can neglect the interference between different avalanches. Hence, the total dissipation rate, $j(\tau)$, at a given time τ, equals the linear superposition of the individual dissipation rates produced by the individual avalanches operating at time τ. Introduce a family of discrete (indicator) functions $\{p^\alpha(\tau)\}_\alpha$. With a fixed α the function $p^\alpha(\tau)$ is equal to unity if an avalanche of type α has been triggered off in the time segment $\tau, \tau + d\tau$, zero otherwise. If we divide the time axis into intervals of length δ, then

$$j(\tau) \;=\; \sum_\alpha \sum_{n=-\infty}^{\infty} f_\alpha(\tau - n\delta) p^\alpha(n\delta) \tag{6}$$

where we have used the convention $f_\alpha(\tau) = 0$ when $\tau < 0$.

Introducing an ensemble of critical systems where each ensemble member is perturbed by adding sand (on randomly chosen sites) with a constant probability rate ν is leading to a transformation of the function $p^\alpha(n\delta)$ into a stochastic process $P^\alpha(n\delta)$.

It turns out that the power spectrum of the stochastic process $J(\tau)$ can be expressed in terms of a weighted average of the power spectra of individual $f_\alpha(\tau)$ signals:

$$S_J(\omega) = \nu \sum_\alpha P(\alpha) |\hat{f}_\alpha(\omega)|^2. \tag{7}$$

$P(\alpha)$ denotes the probability for an avalanche of type α to occur. From this expression we derive the scaling properties of $S_J(\omega)$. First, we assume that it is possible to characterize the individual avalanche signals by the size and lifetime, i.e., $\alpha = (s,t)$. Furthermore, the identity

$$\int_0^t f_{s,t}(\tau) d\tau \;=\; s = \int_0^t \frac{s}{t} f_{1,1}(\frac{\tau}{t}) d\tau \tag{8}$$

suggests the scaling relation

$$f_{s,t}(\tau) = \frac{s}{t} f_{1,1}(\frac{\tau}{t}). \tag{9}$$

Substituting the corresponding result in the frequency domain into Eq. (7) we obtain the fundamental equation for the analysis of the dynamical aspects of the sandpile

$$S_J(\omega) = \nu \sum_s \sum_t |\hat{f}_{1,1}(\omega t)|^2 s^2 P(S = s, T = t). \tag{10}$$

Three different frequency regions have to be considered: A low, an intermediate and a high frequency region. For $\omega \to 0$ the power spectrum becomes white, since the linear superimposed signal $J(\tau)$ cannot contain correlations for times longer than the longest possible lifetime t_{max} of an avalanche, i.e., $S(\omega) \to$ *constant* when $\omega < 1/t_{max}$. In the high frequency region $S_J(\omega) \sim \omega^{\alpha_\infty}$. The value of the exponent α_∞ depends on the specific form of $f_{1,1}(\tau)$. The behavior of $S_J(\omega)$ for intermediate frequencies is determined by $P(S = s, T = t)$. We introduce a weighted lifetime distribution $\Lambda(t)$ by

$$\Lambda(t) = \sum_s s^2 P(S = s, T = t) = E[S^2 | T = t] P(T = t). \tag{11}$$

Assuming that $\Lambda(t)$ exhibits a scaling behavior in an interval $0 < t_1 \leq t \leq t_2 \leq \infty$

$$\Lambda(t) \sim t^{\mu} \tag{12}$$

and is negligible outside an analysis of the scaling behavior of Eq. (10) leads to the result

$$S_J(\omega) \sim \begin{cases} 1, & 0 < -\mu - 1; \\ \omega^{-\mu-1}, & \alpha_\infty < -\mu - 1 < 0; \\ \omega^{\alpha_\infty}, & -\mu - 1 < \alpha_\infty. \end{cases} \tag{13}$$

For $f_{1,1}(\tau)$ equal to a square box function , $\alpha_\infty = -2$, whereas, in the case of a triangular shape of $f_{1,1}(\tau)$, the exponent $\alpha_\infty = -4$. We can prove that $\alpha_\infty \leq -2$. Thus, the only way to obtain a $1/f$ power spectrum is by having a weighted lifetime distribution with an exponent $\mu = 0$, irrespective of the specific form of the superposed signals! Since our simulations results in $\mu > 1.5$ (at least up to 5 dimensions) the power spectrum behaves as $1/f^2$ if the elementary signals are characterized by $\alpha_\infty = -2$. This is also confirmed by a directly numerical measurement of the power spectrum.

3. Conclusion

The connection between the distribution of weighted lifetimes and the corresponding power spectrum for linearly superimposed avalanches was derived. It was shown that the weighted lifetime distribution must be independent of the avalanche lifetime in order to obtain a $1/f$ power spectrum. Furthermore, we examined the temporal behavior of flow of sand down the slope. The power spectrum of the flow were found to behave as $1/f^2$ in all dimensions.

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References

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381, (1987).

[2] For a detailed discussion see J. Stat. Phys., April 1991.