On self-organised criticality in one dimension

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Abstract

In critical phenomena, many of the characteristic features encountered in higher dimensions such as scaling, data collapse and associated critical exponents are also present in one dimension. Likewise for systems displaying self-organised criticality. We show that the one-dimensional Bak-Tang-Wiesenfeld sandpile model, although trivial, does indeed fall into the general framework of self-organised criticality. We also investigate the Oslo ricepile model, driven by adding slope units at the boundary or in the bulk. We determine the critical exponents by measuring the scaling of the kth moment of the avalanche density with system size. The avalanche size exponent depends on the type of drive but the avalanche dimension remains constant.

Key words: self-organised criticality, critical exponents, data collapse.
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Per Bak's passion and insight in science was second to none and his style of work was unique. Per readily identified core problems in science, reduced them to the barest form before solving them. Being a graduate student with Per at Brookhaven National Laboratory and later a collaborator was like attending a never-ending master class. "What the heck, an exponentially decaying function is nothing but a Heaviside function", he told me. We shall shortly see that indeed he is right. Science has lost a great ambassador and I have lost a great personal friend who has influenced my life immensely.

The one-dimensional Bak-Tang-Wiesenfeld (BTW) sandpile model [1] is defined on a discrete lattice consisting of L sites, $i = 1, 2, \ldots, L$. There is a vertical wall at the left boundary next to site $i = 1$ and the pile is open at

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the right boundary next to site $i = L$, where grains can leave the pile. The height, h_i, is the number of grains at column i but it is convenient to refer to local slopes, $z_i = h_i - h_{i+1}$, with $h_{L+1} = 0$. Each site is assigned a critical slope, z_i^c. If the local slope exceeds the critical slope, a grain at the site i will topple to the site $i+1$, that is, $h_i \to h_i - 1$ and $h_{i+1} \to h_{i+1} + 1$. The slopes at neighbouring sites $i \pm 1$ increase by one and may in turn exceed their critical slopes and topple, causing an avalanche that propagates until the condition for a stable configuration with $z_i \le z_i^c$ for all i is reached, see Figure 1.

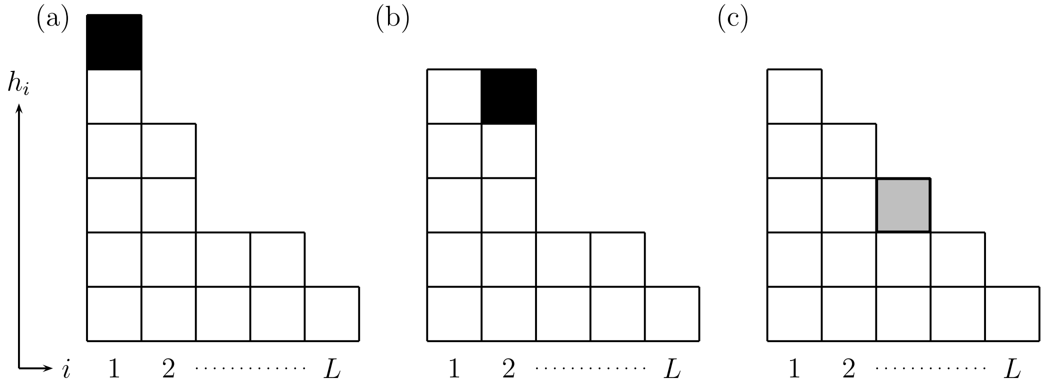

Fig. 1. The $d = 1$ Bak-Tang-Wiesenfeld sandpile model. There is a vertical wall on the left but grains can leave the pile through the open boundary on the right. A grain will topple from the left to the right when the local slope exceeds a critical value, $z_i = h_i - h_{i-1} > z_i^c$. Unstable grains are black while stable grains are white. (a) A grain is added at site $i = 1$ and the critical slope of the first site is exceeded. An avalanche carries the grain onto the second site. (b) The critical slope of the second site is also exceeded and the grain topples to the third site. (c) The grain finally comes to rest on the third site where the critical slope is not exceeded. The avalanche size $s = 2$ is proportional to the energy dissipated.

In the BTW sandpile model, the critical slopes, z_i^c, are constant, independent of position and time and the algorithm for the dynamics is defined as follows.

1. Place the pile in an arbitrary stable configuration with $z_i \le z_i^c$ for all i.
2. Add a grain at a random site i, that is, $h_i \to h_i + 1$.
3. If $z_i > z_i^c$, the site relaxes and

$$\begin{aligned} z_i &\to z_i - 2 \\ z_{i\pm 1} &\to z_{i\pm 1} + 1 \end{aligned} \tag{1a}$$

except when boundary sites topple, where, respectively,

$$\begin{aligned} z_1 &\to z_1 - 2 & \qquad z_L &\to z_L - 1 \\ z_2 &\to z_2 + 1 \quad \text{for } i = 1 & \qquad z_{L-1} &\to z_{L-1} + 1 \quad \text{for } i = L. \end{aligned} \tag{1b}$$

A stable configuration is reached when $z_i \le z_i^c$ for all i.
4. Proceed to step **2.** and reiterate.

Note the separation of time scales which is built into the definition of the model. The addition of grains only takes place when the pile has settled down into a stable configuration. Thus the response of the system (the avalanche) is fast when compared with the interval between additions. A stable configuration, $\mathcal{M}_j$, is either a transient configuration, $\mathcal{T}_j$, or a recurrent configuration, $\mathcal{R}_j$. Transient configurations are not reachable once the system has entered the set of $N_{\mathcal{R}}$ recurrent configurations. When adding a grain to a stable configuration $\mathcal{M}_j$, it evolves into another stable configuration $\mathcal{M}_{j+1}$ by the relaxation rules given above. Symbolically we write $\mathcal{M}_j \mapsto \mathcal{M}_{j+1}$, where the arrow is a shorthand notation for the operation of adding sand and, if necessary, relaxing the pile until it reaches a new stable configuration. Therefore, the index j denotes discrete time steps that are associated with the long time scale of the system. Starting from, say, the empty configuration, which is a transient configuration since it will never be encountered again, we have

$$\underbrace{\mathcal{T}_1 \mapsto \mathcal{T}_2 \mapsto \cdots \mapsto \mathcal{T}_n}_{\text{transient configurations}} \mapsto \underbrace{\mathcal{R}_{n+1} \mapsto \mathcal{R}_{n+2} \mapsto \cdots}_{\text{recurrent configurations}}. \tag{2}$$

After n additions and associated relaxations (if any), the system reaches the set of recurrent configurations, the so-called attractor of the dynamics. For the $d = 1$ BTW sandpile model, $N_{\mathcal{R}} = 1$ since there is only one configuration in the attractor, namely the minimally stable configuration with $z_i = z_i^c \ \forall i$, see Figure 1(c). Any grain introduced into the system after this recurrent configuration has been reached will simply topple down the slope and leave the pile at the open boundary. Note that starting from an empty configuration and adding grains only at $i = 1$, the $d = 1$ BTW sandpile model with $z_i^c = 1$ has $n = L(L + 1)/2$. This is an example of a far more general result [2].

Adding a grain to the pile is equivalent to adding potential energy. This energy is dissipated by the avalanches where, when the threshold for stick-slip behaviour is exceeded, the potential energy is converted to kinetic energy which is dissipated as heat and sound due to the friction between the grains. The critical slope models the friction and causes the stick-slip behaviour. The granular pile may act as a solid when the critical slope is finite. Avalanches are initiated by adding grains of sand to the pile. The total energy dissipated by an avalanche is proportional to the total number of topplings. Therefore, we simply define the avalanche size, s, as the total number of topplings. For a finite system size, L, the pile will eventually reach the set of recurrent configurations. In the attractor of the dynamics, we have a statistically stationary situation where the average number of grains added to the pile equals the average number of grains leaving the pile at the open boundary, or, in terms of energy in the pile, the average energy added equals the average energy dissipated. For the $d = 1$ BTW sandpile model, these statements are correct even for a single addition, without the need for taking statistical averages.

Grains added at random positions i will cause an avalanche of size $s = L - i + 1$. All the avalanches $s = 1, \ldots, L$ are equally probable. Let $P(s, L)$ denote the avalanche size density in a system of size L. The avalanche size density must be normalised, $\sum_{s=1}^{L} P(s, L) = 1$, implying

$$P(s, L) = \begin{cases} \frac{1}{L} & \text{for } 1 \leq s \leq L \\ 0 & \text{otherwise} \end{cases}$$
$$= \frac{1}{L}\, \Theta\left(1 - \frac{s}{L}\right), \tag{3}$$

where we have introduced the Heaviside function $\Theta(x) = 1$ for $x \geq 0$, zero otherwise. Figure 2(a) displays the avalanche size density for system sizes $L = 25, 50, 100$.

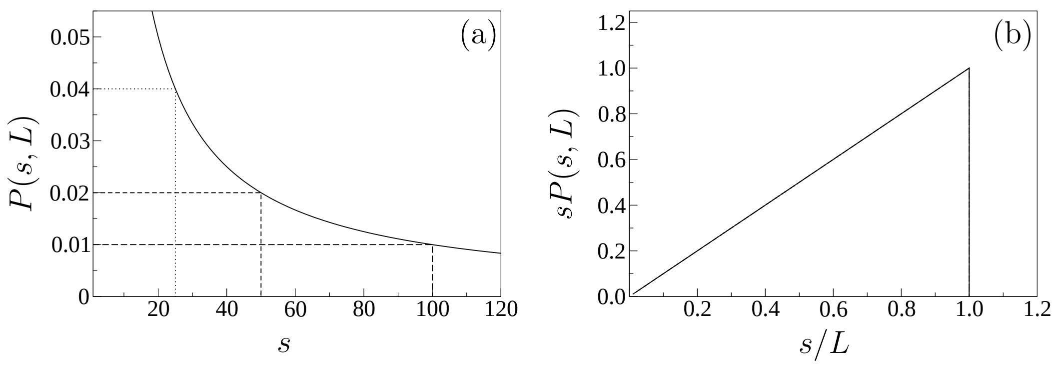

Fig. 2. The avalanche size densities, $P(s, L)$, in the $d = 1$ BTW sandpile model on lattices of linear size $L = 25, 50, 100$ marked with lines of increasing dash length. (a) The avalanche size density, $P(s, L)$, versus avalanche size, s. There is no typical size of an avalanche except for the cutoff avalanche size which increases linearly with system size. The solid line is the graph of s^{-1}. (b) Plotting the rescaled avalanche size density, $sP(s, L)$, versus the rescaled avalanche size, s/L, produces a data collapses onto a universal scaling function $\mathcal{G}_{\mathrm{1d}}^{\mathrm{BTW}}$.

Rewriting the avalanche size density, we find

$$P(s, L) = s^{-1}\, \frac{s}{L}\, \Theta\left(1 - \frac{s}{L}\right) = s^{-1} \mathcal{G}_{\mathrm{1d}}^{\mathrm{BTW}}\left(\frac{s}{L}\right) \tag{4}$$

where we have introduced the scaling function of the avalanche size density, $\mathcal{G}_{\mathrm{1d}}^{\mathrm{BTW}}$, which is a function of the rescaled avalanche size, s/L. Therefore,

$$sP(s, L) = \mathcal{G}_{\mathrm{1d}}^{\mathrm{BTW}}\left(\frac{s}{L}\right). \tag{5}$$

Although the left-hand side of Equation (5) is a function of two variables, s and L, the right-hand side is only a function of one variable s/L. Therefore, all avalanche size densities are identical when suitably transformed and viewed in the appropriate relative scale. For the $d = 1$ BTW sandpile model, the only relevant scale is the cutoff avalanche size L. Therefore, by plotting the rescaled avalanche size density, $sP(s,L)$, versus the rescale avalanche size, s/L, see Figure 2(b), the avalanche size densities fall onto the same curve representing the graph of the scaling function

$$\mathcal{G}_{1d}^{\mathrm{BTW}}(x) = x\Theta(1-x). \tag{6}$$

The kth moment of the avalanche size density can be calculated explicitly,

$$\begin{aligned}
\langle s^k \rangle &= \sum_{s=1}^{\infty} s^k P(s,L) \\
&= \frac{1}{L}\sum_{s=1}^{L} s^k \\
&= \frac{1}{k+1}\sum_{s=1}^{k+1}(-1)^{\delta_{s,k}}\binom{k+1}{s}B_{k+1-s}L^{s-1} \\
&\approx \frac{L^k}{k+1}\left(1+\frac{k+1}{2}L^{-1}\right) \quad \text{for } L \gg 1.
\end{aligned} \tag{7}$$

where $\delta_{s,k}$ is the Kronecker delta, $\binom{k+1}{s}$ the Binomial coefficient and B_l the lth Bernoulli number.

Equation (4) is known as a finite-size scaling ansatz. Why? Well, let us remind ourselves of ordinary critical phenomena. For example, in percolation, the cluster number density at occupation probability p close to the critical occupation probability p_c obeys the scaling ansatz

$$n(s,p) \propto s^{-\tau}\mathcal{G}^{\mathrm{per}}\left(\frac{s}{\xi^D}\right) \quad \text{for } s \gg 1,\, p \to p_c, \tag{8}$$

where D is the fractal dimension, τ is the cluster number density exponent, ξ is the correlation length and $\mathcal{G}^{\mathrm{per}}$ is a scaling function. Equation (8) is valid for infinite system size $L = \infty$. For finite system sizes, the behaviour depends on the relative size of the system to the correlation length. When $L \gg \xi$ the system is effectively infinite in size and Equation (8) applies. However, when $L \ll \xi$ (which is the case for all system sizes at the critical point since $\xi = \infty$) we have to replace Equation (8) with the finite-size scaling ansatz

$$n(s,p,L) \propto s^{-\tau}\tilde{\mathcal{G}}^{\mathrm{per}}\left(\frac{s}{L^D}\right) \quad \text{for } s \gg 1, p \to p_c, 1 \ll L \ll \xi, \tag{9}$$

with the same critical exponents but another scaling function $\tilde{\mathcal{G}}^{\mathrm{per}}$ which depends on, for example, geometry and boundary conditions. The finite-size scaling ansatz Equation (9) is the analogue to Equation (4). However, the $d = 1$ BTW sandpile model has by itself organised into a configuration at the critical point without any fine-tuning of external parameters. The model displays self-organised criticality [3,4].

In general, for higher dimensions, we might expect the avalanche size density to satisfy the scaling ansatz

$$P(s,L) \propto s^{-\tau}\mathcal{G}\left(\frac{s}{L^D}\right) \quad s \gg 1, L \gg 1, \tag{10}$$

where D is the avalanche dimension, τ is the avalanche size exponent, and $\mathcal{G}$ is a scaling function which for small and large arguments

$$\mathcal{G}(x) \propto \begin{cases} \mathcal{G}(0) + \mathcal{G}'(0)x + \frac{1}{2}\mathcal{G}''(0)x^2 + \cdots & \text{for } x \ll 1 \\ \text{decay rapidly} & \text{for } x \gg 1. \end{cases} \tag{11}$$

In summary, there is only one recurrent configuration for the $d = 1$ BTW sandpile model. The avalanche size density satisfies the scaling ansatz Equation (4) with the scaling function $\mathcal{G}_{1\mathrm{d}}^{\mathrm{BTW}}(x) = x\Theta(1-x)$ and critical exponents $D = 1$ and $\tau = 1$. These results are in line with equilibrium critical systems in the lower critical dimension. For example, in $d = 1$ percolation, $p_c = 1$, and there is only one configuration where with all sites (bonds) are occupied. In the $d = 1$ Ising model, $(T_c, H_c) = (0, 0)$, and all spins are aligned. Nevertheless, these one-dimensional models have the fingerprint of being critical since their behaviour is consistent with the general framework of scaling, data collapse and associated critical exponents.

It is also interesting to note that the scaling function for the $d = 1$ BTW sandpile model vanishes for zero argument, $\mathcal{G}_{1\mathrm{d}}^{\mathrm{BTW}}(0) = 0$ as does, for example, the scaling function in $d = 1$ percolation, $\mathcal{G}_{1\mathrm{d}}^{\mathrm{per}}(x) = x^2 \exp(-x)$. However, in higher dimension, typically $\mathcal{G}(0) \neq 0$. For example, the mean-field theory of sandpile models yield an exponentially decaying function.

Furthermore, the results of the $d = 1$ BTW sandpile model highlight that the avalanche size exponent, τ, does not relate to the decay of the avalanche size density, as is often mistakenly argued, but rather to the decay of the distinctive features of the avalanche size density, in this case the upper right corner where the cutoff sets in, see solid line in Figure 2(a). When attempting to make a data collapse, the first operation of multiplying the avalanche size densities with s^τ is to make sure that all the distinctive features of each graph all lie at the same vertical position with only the horizontal positions of this feature differentiating the graphs. Rescaling the avalanche size, s, with the factor L^D will in turn make the horizontal positions collapse.

Assuming Equation (10) is valid for all avalanche sizes s we can calculate the kth moment of the avalanche size density by approximating the sum with an integral

$$\begin{aligned}
\langle s^k \rangle &\approx \int_1^\infty s^{k-\tau} \mathcal{G}\left(\frac{s}{L^D}\right) ds \\
&= \int_{1/L^D}^\infty (xL^D)^{k-\tau} \mathcal{G}(x) L^D \, dx \qquad x = s/L^D \\
&= L^{D(k+1-\tau)} \int_{1/L^D}^\infty x^{k-\tau} \mathcal{G}(x) \, dx. \qquad (12)
\end{aligned}$$

In the limit $L \gg 1$, the lower limit of the integral tends to zero. The integral is definite and just a number. Therefore, if the avalanche size density satisfies the scaling ansatz Equation (10), the kth moment of the avalanche size

$$\langle s^k \rangle \propto L^{D(k+1-\tau)} \quad \text{for } L \gg 1. \qquad (13)$$

In addition, if the scaling function is known, we can extract the proportionality constant to the leading order and the order of the correction to scaling. For example, in $d = 1$, where $D = 1, \tau = 1$, and $\mathcal{G}_{\mathrm{1d}}^{\mathrm{BTW}}(x) = x\Theta(1-x)$ we find

$$\langle s^k \rangle \approx L^k \int_{1/L}^\infty x^{k-1} \mathcal{G}_{\mathrm{1d}}^{\mathrm{BTW}}(x) \, dx = L^k \int_{1/L}^1 x^n \, dx = \frac{L^k}{k+1} \left(1 - L^{-1}\right) \qquad (14)$$

consistent with the exact result in Equation (7), except for the proportionality constant of the order of correction to scaling.

The $d = 1$ Oslo ricepile model [5] is similar to the BTW sandpile model, except that grains are only added at site $i = 1$ and the critical slopes are neither constant along the pile nor in time. The critical slope, z_i^c, is chosen randomly every time site i topples. Therefore, the model mimics the experiment on a ricepile where grains were added close to the vertical wall. It was observed that the slope varies along the profile as well as in time at a given site [6]. In general, we can use any discrete probability distribution of critical slopes. However, we will consider the simplest case where the critical slope, z_i^c, is chosen at random, between 1 and 2 when site i topples. Note that the Oslo model has been mapped into driven interfaces [8,9] and Dhar has recently developed an operator algebra for the Oslo model [2].

The Oslo model has a set of $N_{\mathcal{R}} = (aG^L + a^{-1}G^{-L})/\sqrt{5}$, $G = (3+\sqrt{5})/2, a = (1+\sqrt{5})/2$ recurrent configurations [7]. In the attractor, there is no typical scale of the avalanche size other than that set by the system size. The $d = 1$ Oslo ricepile model displays self-organised criticality and the avalanche size density satisfies the scaling ansatz Equation (10) with non-trivial critical exponents. The critical exponents can be determined by collapsing the data of the avalanche size density for various system sizes, see Figure 3.

Equation (13) suggests an alternative and often superior way to determine the critical exponents. First, one measures the kth moments of the avalanche size

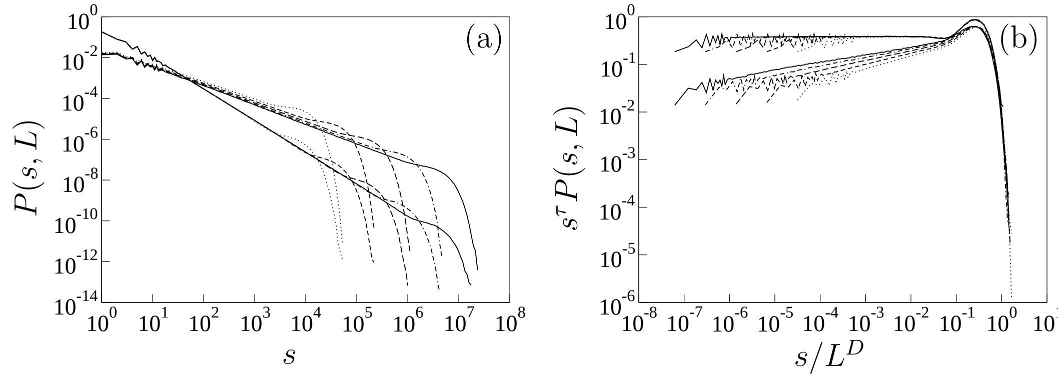

Fig. 3. (a) The avalanche size densities, $P(s, L)$, in the $d = 1$ Oslo ricepile model (boundary and bulk driven) for lattice sizes $L = 100, 200, 400, 800, 1600$ versus avalanche size, s. There is no typical size of an avalanche except for the cutoff avalanche size which increases with system size. (b) Plotting the rescaled avalanche size density, $s^\tau P(s, L)$, versus the rescaled avalanche size, s/L^D, produces a data collapses onto a universal scaling function $\mathcal{G}^{\mathrm{Oslo}}$ when using $\tau = 1.55$, $D = 2.25$ (boundary driven) and $\tau = 1.11$, $D = 2.25$ (bulk driven).

density as a function of system size, see Figure 4(a) and then one extracts the exponent describing the scaling of the kth moment with system size

$$\frac{d \log \langle s^k \rangle}{d \log L} = D(k + 1 - \tau) \quad \text{for } L \gg 1. \tag{15}$$

Second, by plotting this quantity as a function of the moment k, we can determine D and hence τ, see Figure 4(b). Numerically, for the boundary driven Oslo model we find $D = 2.2496(12) \stackrel{?}{=} \frac{9}{4}$ and $\tau = 1.5555(2) \stackrel{?}{=} \frac{14}{9}$. The fractional values suggested for the avalanche dimension and the avalanche size exponent are consistent with the scaling relation $D(2 - \tau) = 1$ that follows from Equation (13) and the analytic result of $\langle s \rangle = L$.

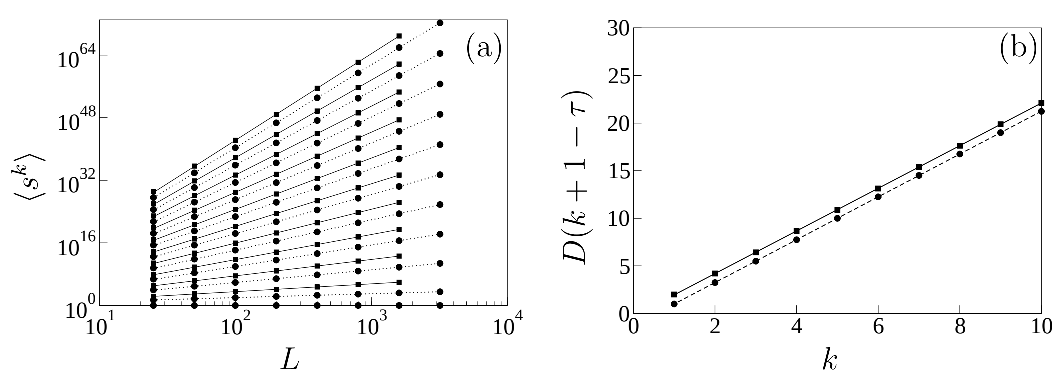

Fig. 4. (a) The kth moment $\langle s^k \rangle$, $k = 1, 2, \ldots, 10$ as a function of system size, L, for boundary driven (solid circles) and bulk driven (solid squares) Oslo model. System of sizes $L = 25, 50$ are excluded when extracting the slope $d \log \langle s^k \rangle / d \log L$. (b) The measured exponent $D(k+1-\tau)$ as a function of moment k. For the boundary driven model (solid circles), the slope $D = 2.2496(12)$ implying $\tau = 1.5555(2)$, while the bulk drive model (solid squares) yields a slope $D = 2.24(2)$ implying $\tau = 1.11(2)$.

When changing the Oslo ricepile model from adding a grain to the boundary to adding *a slope unit* to a randomly chosen site i, that is, $z_i \to z_i + 1$, it has been shown analytically that $\langle s \rangle = L^2/3 + L/2 + 1/6$ [10]. Therefore, the critical exponents must satisfy $D(2-\tau) = 2$. The normal procedure of determining the exponents by data collapse of the avalanche size densities is difficult to apply, see Figure 3(a) and (b). $D = 2.25$ and $\tau = 1.11$ produces a reasonable collapse for large values of s. Note again, that the avalanche size exponent $\tau = 1.11$ does not reflect the decay of the avalanche size density. It is determined purely by the making sure that all the distinctive features of each graph, namely where the cutoff sets in, all lie at the same vertical position. But using the method of measuring the scaling of the kth moments of the avalanche size densities with system size outlined above, we find $D = 2.24(2) \stackrel{?}{=} \frac{9}{4}$ and $\tau = 1.11(2) \stackrel{?}{=} \frac{10}{9}$. It is intriguing to note that the avalanche dimension remains constant while the avalanche size exponent changes in order to satisfy the relevant scaling relation. One might speculate whether D is a more fundamental exponent in a hierarchy of critical exponents associated with self-organised critical behaviour.

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