EUROPHYSICS LETTERS 15 April 2000

*Europhys. Lett.*, **50** (2), pp. 162–168 (2000)

# On the physical relevance of extremal dynamics

A. S. Datta [1](*), K. Christensen [2](**) and H. Jeldtoft Jensen [1](***)

[1] *Department of Mathematics, Imperial College - 180 Queens Gate London SW7 2BZ, UK*

[2] *Blackett Laboratory, Imperial College - Prince Consort Road London SW7 2BZ, UK*

(received 5 November 1999; accepted in final form 9 February 2000)

PACS. `05.65+b` – Self-organised systems.
PACS. `05.45-a` – Nonlinear dynamics and nonlinear dynamical systems.
PACS. `87.10+e` – General theory and mathematical aspects.

**Abstract.** – Taking the Bak-Sneppen evolution model as an example (*Phys. Rev. Lett.*, **71** (1993) 4083), we study extremal dynamics where $M > 1$ smallest barriers are simultaneously updated as opposed to models in the limit where only the smallest barrier is updated. We investigate the scaling properties of the nearest-neighbour and the random-neighbour model. We demonstrate that the behaviour of models with extremal dynamics in the limit of single update per time step is irrelevant to physical observations.

*Introduction.* – Extremal dynamics is used to model the temporal evolution of many different systems. Invasion percolation is one of the most prominent examples [1] but attempts have been also made to catch some of the salient features of biological evolution in models involving extremal dynamics [2, 3] as well as in some other models related to self-organised criticality [4, 5]. In extremal models, the dynamics consist of a global search for the site in the system with the smallest (or largest) value of the dynamical variable. This site and its neighbours are then updated according to the specific algorithm of the model considered, whereupon the procedure is repeated.

To be specific, let us consider the particular case of invasion percolation. As pointed out by Wilkinson and Willemsen, the procedure of updating only the link with smallest resistance corresponds to the limit of zero flux [1]. From this perspective it appears immediately relevant to study the behaviour as one approaches the limit of slow but finite drive, that is, study the system when the sites with the $M$ smallest values of the dynamical variable are updated simultaneously.

It has previously been tacitly assumed, that the smallest and the second smallest dynamical variables are well separated, in which case the single update limit is appropriate. We find, however, that the probability density of the separation between the smallest and second smallest dynamical variables has its maximum at zero separation. Thus one will inevitably

(*) E-mail: `a.s.datta@ic.ac.uk`
(**) E-mail: `k.christensen@ic.ac.uk`
(***) E-mail: `h.jensen@ic.ac.uk`

have to consider the evolution of the second smallest dynamical variable as well. Clearly, if this is a relevant perturbation, then updating the $M > 2$ smallest barriers must lead to even stronger effects than updating the $M = 2$ smallest barriers. Thus, for simplicity, we restrict our discussion to the $M = 2$ case.

We study the scaling properties of the nearest-neighbour version of the Bak-Sneppen model in one dimension as well as the random-neighbour version, and we pay special attention to whether the data of the avalanche durations are described by simple or multifractal scaling. For both models, we find that updating the two smallest ($M = 2$) rather than just the smallest variable has surprisingly significant effects.

In the critical random neighbour model, the ultimate *asymptotic* scaling behaviour is identical for the two cases $M = 1$ and $M = 2$. For the regime accessible in the simulations, simple scaling works well for $M = 1$ while multifractal scaling seems more appropriate for $M = 2$ when approaching the critical point. We attribute this finding to a dramatic decrease, with increasing $M$, in the size of the scaling regime for which the asymptotic behaviour at the critical point can be probed. It is quite alarming that with a reduced variable as small as $10^{-3}$, the asymptotic behaviour of the critical point cannot be probed. Thus the limit of single update per time step of models with extremal dynamics is irrelevant to physical observations even though the simple scaling would be recovered were it possible to approach the critical point with *unlimited* resolution.

In the nearest-neighbour version, a similar scenario is observed. In addition, for $M = 1$ it is not possible to extract an unambiguous scaling exponent $\tau$. The value $\tau \approx 1.07$ for $M = 1$ [5, 6] is observed only right at the critical point. We cannot exclude that the ultimate *asymptotic* forms of the probability densities of avalanche durations at the critical point are identical for $M = 1$ and $M = 2$.

However, from the simulations we conclude that the asymptotic limit of the $M = 2$ case is of little relevance to experiments like invasion percolation of fluids into porous media [7]. We expect that significant deviations from the asymptotic critical behaviour may be dominating experimentally observed data for systems which are related to extremal models.

*The model.* – Let us recapitulate the definition of the Bak-Sneppen model of biological evolution. Consider a lattice in $d$ dimensions consisting of $L$ sites with periodic boundary conditions. To each lattice point (species) $x$ is ascribed a random variable $B(x) \in [0, 1)$ (the fitness). A time step consists of locating the site $x_{\min}$ with the minimum value of $B(x)$. The site $x_{\min}$ and its nearest neighbours are updated by replacing the $B$-value on each of these sites by a new random number drawn uniformly from the interval $[0, 1)$. In the language of biological evolution, the least fit species is most likely to undergo a mutation. However, that will change the fitness landscape of the species with which it interacts, and they are also assigned new (random) fitnesses. In the random-neighbour version of the model, the site with the smallest $B$-value is refreshed together with $K - 1$ new distinct randomly chosen "neighbour" sites, that is, the neighbourhood is annealed and chosen anew for every time step.

We generalise the model by locating the sites with the $M$ smallest $B$ values and their nearest-neighbour sites. The $B$-values on these sites are refreshed. In the random-neighbour version we locate the $M$ smallest $B$-values and refresh these together with $K - M$ distinct randomly chosen "neighbour" sites, that is, $K$ sites are assigned *new* $B$-values.

The *minimum* $B$ value in each time step is called $B_{\min}(t)$. Avalanches —also called forward avalanches— are defined with respect to a threshold $B_0$ [2]. An avalanche starts at time $t = 1$ if $B_{\min}(t = 0) > B_0$ and $B_{\min}(t = 1) < B_0$. The avalanche is of duration $T$ if $B_{\min}(t)$ goes above $B_0$ for the first time when $t = T$.

*Results. Random-neighbour model.* – The random-neighbour model is used as a reference in order to discuss extremal models with multiple updates per time step. The mean-field theory of the probability density $p(B)$ of the site variables, which is exact for the random-neighbour model, is given in sect. 5.2.5 in ref. [8] for the general case $K > M \geq 1$ and ref. [9] for the $M = 1$ case. The important point is that $p(B)$ in the limit $L \to \infty$ is a step function:

$$p(B) = \begin{cases} 0, & \text{for } 0 \leq B < B_\mathrm{c} = M/K, \\ 1/(1-B_\mathrm{c}), & \text{for } B_\mathrm{c} < B \leq 1. \end{cases} \tag{1}$$

This allows us to simulate the $L = \infty$ limit, thus eliminating finite system size effects and highlighting the effect of changing $M = 1$ to $M = 2$. We note that the dynamics of the process is controlled by the instantaneous number of active sites $n_\mathrm{act}(t)$. The temporal evolution of the number of active sites in an avalanche is represented in the following procedure:

1. To initiate an avalanche, $n_\mathrm{act}(t = 0) = M$
2. $n_\mathrm{act}(t + 1) = \mathrm{Max}\{n_\mathrm{act}(t) - M, 0\}$
3. Do $\quad i = 1, K$
   $n_\mathrm{act}(t + 1) = n_\mathrm{act}(t + 1) + 1$ with probability $B_0$
   End Do
4. If $\quad n_{act}(t + 1) > 0$ Then
   $t = t + 1$
   Goto 2
   Else
   Avalanche has terminated.
   End If

Thus we create $K$ new sites, each of them is active with probability $B_0$. Hence, on average, we create $KB_0$ new *active* sites. Accordingly, the average branching ratio is $KB_0/M$. As $B_0 \to B_\mathrm{c} = M/K$, the branching ratio tends to the critical value 1. In this limit, the probability density of avalanche durations is given by the simple scaling ansatz [10, 11]

$$P_{B_0}(T) = T^{-\tau} f(T(B_\mathrm{c} - B_0)^{1/\sigma}), \tag{2}$$

where $\tau = 3/2$ and $\sigma = 1/2$ are critical scaling indices, and $f$ the scaling function.

We concentrated here on the two cases $(M, K) = (1, 2)$, the single update limit, and $(M, K) = (2, 3)$ representing a system with multiple updates per time step. Figure 1(a) displays the probability density of the avalanche durations using the simple scaling ansatz in eq. (2). With the proper values $\tau = 3/2$ and $\sigma = 1/2$, the simple scaling ansatz produces a data collapse for $M = 1$ but not for $M = 2$. In fig. 1(b), we have used a multifractal scaling ansatz of the form

$$\begin{aligned} &\log(P_{B_0}(T))/\log((B_\mathrm{c} - B_0)^{-1}/t^\star) = \\ &\qquad g(\log(T/T^\star)/\log((B_\mathrm{c} - B_0)^{-1}/t^\star)), \end{aligned} \tag{3}$$

where $t^\star$ and $T^\star$ are constants [12]. The argument is often denoted $\alpha = \log(T/T^\star)/\log((B_\mathrm{c} - B_0)^{-1}/t^\star)$, and the local slope $\mathrm{d}g/\mathrm{d}\alpha$ determines the spectrum of scaling indices. If $g$ is a linear function, the local slope is constant, and the multifractal scaling ansatz would reduce to a simple scaling ansatz. This is what happens when $M = 1$. In the case $M = 2$, both the

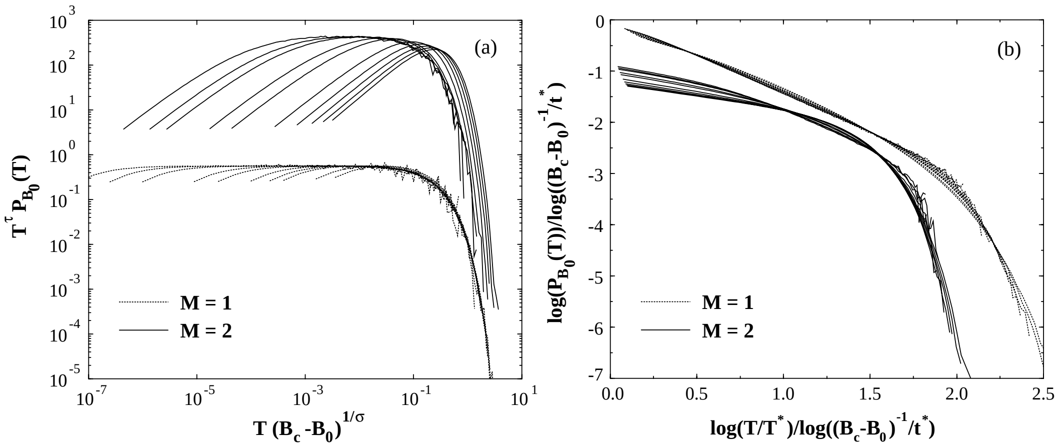


Fig. 1 – The measured probability density of avalanche durations in the random-neighbour version of the generalised Bak-Sneppen model for $M = 1$ and $M = 2$ with the reduced avalanche threshold $b = (B_c - B_0)/B_c \in [10^{-3}, 10^{-1}]$. (a) Simple scaling ansatz eq. (2) using $\tau = 3/2$ and $\sigma = 1/2$. $M = 1$ (dotted lines) with $B_c = 1/2$. $M = 2$ (solid lines, data shifted upwards by a factor 100) with $B_c = 2/3$. (b) Multifractal scaling ansatz eq. (3). $M = 1$ (dotted lines) with $T^\star = 0.5, t^\star = 1.0$, and $B_c = 1/2$. $M = 2$ (solid lines, data shifted downwards by subtracting 0.5) with $T^\star = 0.7, t^\star = 0.5$, and $B_c = 2/3$.

simple and multifractal scaling forms exhibit no clear collapse of the data. Rather, the plotted data appears to be tending towards an asymptotic scaling form. Hence, although a scaling collapse may occur in the limit $b = (B_c - B_0)/B_c \to 0$, this regime is clearly not reached for $M = 2$. Note that the reduced avalanche threshold $b \in [10^{-3}, 10^{-1}]$ for the data shown. Thus, for $M = 2$, the true scaling regime is not accessible.

*Results. Nearest-neighbour model.* – The difference between $M = 1$ and $M = 2$ is even greater in the nearest-neighbour model. We are, however, not able to solve this model analytically and have to rely on simulations. All the reported results are obtained with $10^{10}$ time steps and $L = 10^5$. No single avalanche ever covered the whole system. Figure 2(a) contains a simple scaling plot of the probability density $P_{B_0}(T)$ for $M = 1$ and $M = 2$. Note that different $\tau$ exponents have been used in order to obtain a constant scaling function for small arguments. We find that the avalanche exponent has apparently changed from the value $\tau$ around 1 for the $M = 1$ case to $\tau \approx 0.8$ for the $M = 2$ case. The fact that $\tau(M = 2) < 1$ and also $\tau(M = 1) \approx 0.98 < 1$ makes it important to include the normalisation factor in the simple scaling plots of $P_{B_0}(T)$. Figure 2(b) uses the multifractal scaling ansatz in eq. (3) which is more suited to fitting our data, especially for the $M = 2$ case.

From the simple scaling ansatz with $M = 1$, we also notice that the critical exponent changes systematically as the critical point is approached. Assuming a "correction to the scaling exponent" of the form

$$\tau(B_0) = \tau(B_c) - a(B_c - B_0)^{\epsilon}, \tag{4}$$

when $B \to B_c$, the value $\tau(B_c) = 1.074$ can be extracted from the forward avalanche data, see fig. 3.

The elegant scaling relation $\tau_b^{\text{all}} = 3 - \tau$ [5,13] between the exponent $\tau_b^{\text{all}}$ for the probability density of all backward avalanches (for a definition, see, *e.g.*, ref. [13]) and the (forward)

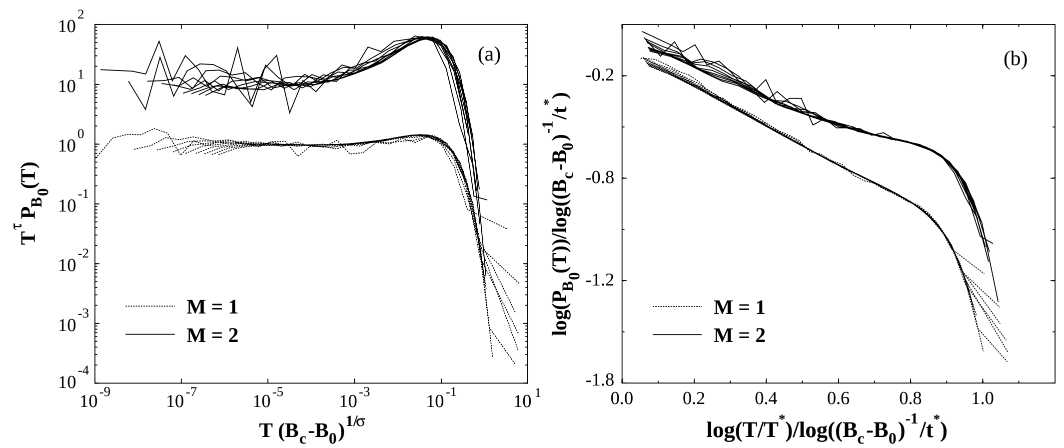


Fig. 2 – The probability density of avalanche durations in the nearest-neighbour version of the generalised Bak-Sneppen model for $M = 1$ with the reduced avalanche threshold $b \in [1.5 \cdot 10^{-3}, 1.5 \cdot 10^{-2}]$ and $M = 2$ with $b \in [2.7 \cdot 10^{-3}, 1.5 \cdot 10^{-2}]$. (a) Simple scaling ansatz, $M = 1$ (dotted lines) with $\tau = 0.98, 1/\sigma = 3.0$, and $B_{\mathrm{c}} = 0.66702$. $M = 2$ (solid lines, data shifted upwards by a factor 25) with $\tau = 0.8, 1/\sigma = 3.2$, and $B_{\mathrm{c}} = 0.6347$. (b) Multifractal scaling ansatz, $M = 1$ (dotted lines) with $T^\star = 0.3, t^\star = 0.3$ and $B_{\mathrm{c}} = 0.66702$. $M = 2$ (solid lines, data shifted upwards by adding 0.25) with $T^\star = 0.3, t^\star = 0.3$ and $B_{\mathrm{c}} = 0.6347$.

avalanches exponent $\tau$ only holds when the following two conditions are satisfied: a) $\tau$ must be greater than 1, and b) the conditional backward avalanche exponent $\tau_{\mathrm{b}}$ must be less than 1. As we have seen, it is not possible from the probability density of (forward) avalanches to *directly* establish $\tau > 1$ even for $M = 1$. This makes it problematic to determine $\tau$ by use of the relation $\tau = 3 - \tau_{\mathrm{b}}^{\mathrm{all}}$, since for all accessible $B_0$, the effective $\tau(B_0) < 1$. The asymptotic

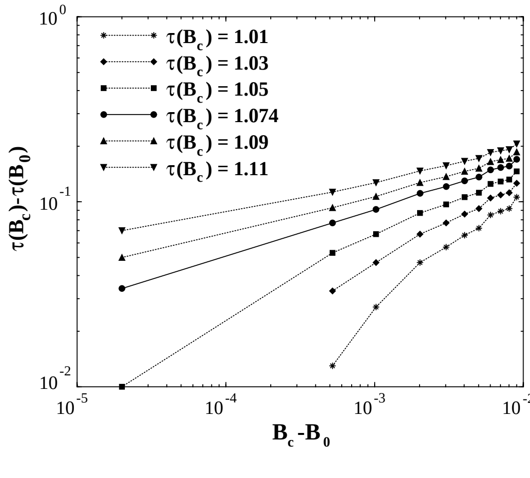


Fig. 3 – Correction to the scaling exponent applied to extract the avalanche size exponent $\tau(B_{\mathrm{c}})$ in the nearest-neighbour model with $M = 1, B_{\mathrm{c}} = 0.66702$. A similar procedure for the $M = 2$ case could not be implemented.

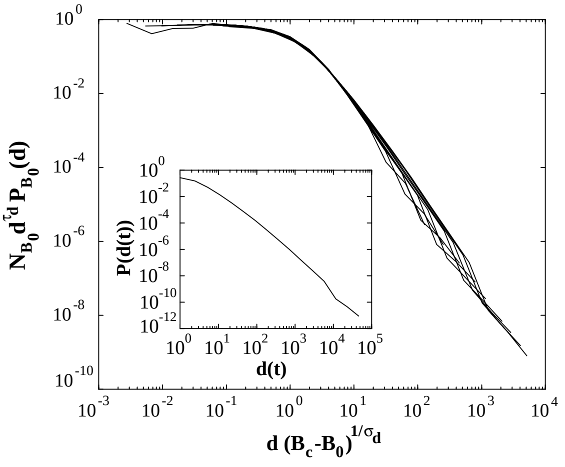


Fig. 4 – A simple scaling plot of the probability density of the linear distance $d(t=0)$ between the positions of the two smallest sites when a $B_0$ avalanche is initiated in the nearest-neighbour model with $M = 2, \tau_d = 0.1, 1/\sigma_d = 1.1$, and $B_c = 0.6347$. The inset displays the probability density of the linear distance $d(t)$.

value of $\tau(B_c) > 1$ can, however, be extracted by the *indirect* procedure described in relation to fig. 3.

It is important to mention that the update of the two smallest $B$-values is correlated. Let $d(t) = \mid x_{\min_1}(t) - x_{\min_2}(t) \mid$ denote the linear distance between the positions of the two smallest sites at time $t$. The probability density of the distance between the two smallest sites $d(t = 0)$ when a $B_0$ avalanche is initiated is flat with a crossover to a power law decay with exponent 2.7, see fig. 4. In the double limit $L \to \infty$ and $B_0 \to B_c$, the probability density of the distance $d(t = 0)$ between the two smallest sites is uniform. However, that is experimentally irrelevant. The relevant case $L < \infty$ and $B_0 \to B_c^-$ will always exhibit correlations between the two initial updates, as displayed in fig. 4. The reason for these correlations is that the two initial updates of a $B_0 < B_c$ avalanche can, due to the hierarchical structure of the avalanches, be considered as an ordinary update of a $B_0^*$ avalanche with $B_0 < B_0^* < B_c$. This also explains why the power law at large value of $d(t = 0)$ is the same as the power law for probability density of the linear distance $d(t) = \mid x_{\min_1}(t) - x_{\min_2}(t) \mid$ between the positions of the smallest and the second smallest $B$-value, see insert in fig. 4.

The above discussion makes it clear that, in a finite system, one cannot just consider the avalanches produced in the $M = 2$ case as a linear random superposition of simultaneously evolving $M = 1$ avalanches. The reason is that updates of the second smallest $B$-value may involve a site with a value of $B > B_0$. That happens whenever the number of active sites equals one. Such an update can produce new active sites disconnected from the sites hitherto touched by the present avalanche.

*Conclusion.* – The physical relevant case, $M > 1$, is unrelated to the asymptotic behaviour of the $M = 1$ case, even in the limit $L \to \infty$, rendering the single update completely irrelevant for physical realisations. Generalising, we have shown that the asymptotic critical behaviour of extremal models is inaccessible as soon as more than the absolute extremal site is allowed to evolve in each time step. Hence, experiments will not be able to see the scaling exponents of the critical point but rather some effective noncritical behaviour. Concretely, we expect that experiments on invasion percolation will measure an effective avalanche exponent that will deviate significantly from the value of the ideal theoretical extremal model.

* * *

We are grateful to H. GILHØJ for collaboration during the early stages of the present project. HJJ is supported by EPSRC under contract GR/L12325. KC was supported from the European Union Human Capital and Mobility Programme contract number ERBFM-BICT961215.

REFERENCES


[1] WILKINSON D. and WILLEMSEN J. F., *J. Phys. A*, **16** (1983) 3365.
[2] BAK P. and SNEPPEN K., *Phys. Rev. Lett.*, **71** (1993) 4083.
[3] FLYVBJERG H., SNEPPEN K. and BAK P., *Phys. Rev. Lett.*, **71** (1993) 4087.
[4] BAK P., *How Nature Works* (Oxford University Press) 1997.
[5] PACZUSKI M., MASLOV S. and BAK P., *Phys. Rev. E*, **53** (1996) 414.
[6] GRASSBERGER P., *Phys. Lett. A*, **200** (1995) 277.
[7] FRETTE V., FEDER J., JØSSANG T., MEAKIN P. and MÅLØY K. J., *Phys. Rev. E*, **50** (1994) 2881.
[8] JENSEN H. J., *Self-Organized Criticality* (Cambridge University Press) 1998.
[9] DE BOER J., DERRIDA B., FLYVBJERG H., JACKSON A. D. and WETTIG T., *Phys. Rev. Lett.*, **73** (1994) 906.
[10] CHRISTENSEN K. and OLAMI Z., *Phys. Rev. E*, **48** (1993) 3361.
[11] HARRIS T. E., *The Theory of Branching Processes* (Springer-Verlag, Berlin) 1963.
[12] KADANOFF L., NAGEL S., WU L. and ZHOU S., *Phys. Rev. A*, **39** (1989) 6524.
[13] MASLOV S., *Phys. Rev. Lett.*, **74** (1995) 562.