Self-Organized Criticality in a Continuous, Nonconservative Cellular Automaton Modeling Earthquakes

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We introduce a new nonconservative self-organized critical model. This model is equivalent to a quasistatic two-dimensional version of the Burridge-Knopoff spring-block model of earthquakes. Our model displays a robust power-law behavior. The exponent is not universal; rather it depends on the level of conservation. A dynamical phase transition from localized to nonlocalized behavior is seen as the level of conservation is increased. The model gives a good prediction of the Gutenberg-Richter law and an explanation to the variances in the observed b values.

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The dynamics of earthquake faults may provide a physical realization of the recently proposed idea of self-organized criticality (SOC). Bak, Tang, and Wiesenfeld (BTW) introduced the concept of self-organized criticality: Dynamical many-body systems reach a critical state without the need to fine-tune the system parameters [1]. BTW showed that a certain class of systems drive themselves into a statistically stationary state characterized by spatial and temporal correlation functions exhibiting power-law behavior. Hence, the system has no intrinsic length or time scale and is in this sense critical. The study of the SOC systems has to a great extent been based on simulations using cellular automaton models. The majority of these simulations have been limited to *conservative* models. It has been suggested that the necessary (and sufficient) condition for SOC is indeed a conservation law [2,3]. This seems to be the situation for SOC models where perturbation is done locally as in the original BTW model [4]. Recently, though, it was shown that a special nonconservative model with a global perturbation displays SOC [5].

Earthquakes are probably the most relevant paradigm of self-organized criticality. In 1956 Gutenberg and Richter realized that the rate of occurrence of earthquakes of magnitude M greater than m is given by the relation

$$\log_{10} N(M>m)=a-bm\,. \qquad (1)$$

This is the Gutenberg-Richter law [6]. The parameter b has been recorded to have a wide range of values for different faults. Findings of b from 0.80 to 1.06 for small earthquakes and 1.23 to 1.54 for large earthquakes have been reported [7].

The energy (seismic moment) E released during the earthquake is believed to increase exponentially with the earthquake magnitude,

$$\log_{10} E=c+dm\,, \qquad (2)$$

where the parameter d is 1 and $\frac{3}{2}$ for small and large earthquakes, respectively [8]. Thus the Gutenberg-Richter law is transformed into a power law for the number of observed earthquakes with energy greater than E,

$$N(E_0>E)\sim E^{-b/d}=E^{-B}\,. \qquad (3)$$

Note that B is in the *same* range for both small and large earthquakes, namely, 0.80–1.05.

Bak and Tang indicated that the simple conservative SOC models can serve as a framework for explaining the power-law behavior, giving a B value of 0.2 [9]. Similar results are obtained for two-dimensional models in [10,11]. Otsuka was the first to simulate a 2D version of the Burridge-Knopoff model and he found $B\approx 0.8$ [12]. Carlson and Langer proposed a 1D dynamical version of the Burridge-Knopoff model [13–15]. A similar quasistatic model in one dimension was investigated by Nakanishi [16,17].

We introduce a generalized, continuous, nonconservative cellular automaton model that displays SOC [18]. This model has several interesting aspects. First, it is directly mapped into a two-dimensional version of the famous Burridge-Knopoff spring-block model for earthquakes. Second, it displays a robust SOC behavior over a very wide range of conservation levels. Third, we find that the level of conservation has an impact on the power laws obtained. We see a transition from localized behavior into nonlocalized behavior as the level of conservation is increased. Finally, the dependence of the power laws on the conservation allows us to explain the wide variances in the Gutenberg-Richter law as a result of the variances of the elastic parameters.

The Burridge-Knopoff spring-block model is a two-dimensional dynamical system of blocks interconnected by springs. Each block is connected to the four nearest neighbors. Additionally, each block is connected to a single rigid driving plate by another set of springs as well as connected frictionally to a fixed rigid plate [see Fig. 1(a)]. The blocks are driven by the relative movement of the two rigid plates. When the force on one of the blocks is larger than some threshold value F_{th} (the maximal static friction), the block slips. We assume that the moving block will slip to the zero-force position. This assumption is not essential for the behavior of the model as will become evident later on. The slip of one block will redefine the forces on its nearest neighbors. This can result in further slips and a chain reaction can evolve.

For the purpose of mapping the spring-block model into a cellular automaton model we define an $L\times L$ array of blocks by (i,j), where i,j are integers restricted to the

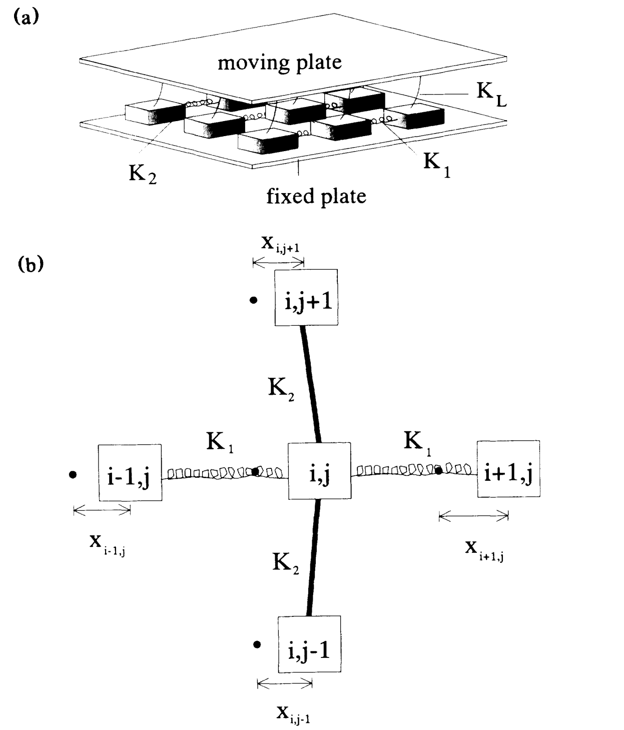

FIG. 1. The geometry of the spring-block model. (a) The two-dimensional system of blocks connected by springs. The strain of the blocks increases uniformly as a response to the relative movement of the rigid plates. (b) A detailed picture of a given block (i,j) and its surroundings.

interval between 1 and L. The displacement of each block from its relaxed position on the lattice is defined as $dx_{i,j}$. The total force exerted by the springs on a given block (i,j) is expressed by

$$F_{i,j}=K_1[2dx_{i,j}-dx_{i-1,j}-dx_{i+1,j}] \\ +K_2[2dx_{i,j}-dx_{i,j-1}-dx_{i,j+1}]+K_L dx_{i,j}\,, \qquad (4)$$

where K_1, K_2, and K_L denote the elastic constants [see Fig. 1(b)].

When the two rigid plates move relative to each other the total force on each block increases uniformly (with a rate proportional to $K_L V$, where V is the relative velocity between the two rigid plates) until one site reaches the threshold value and the process of relaxation begins (an earthquake is triggered). It can easily be shown that the redistribution of strain after a local slip at the position (i,j) is given by the relation

$$F_{i\pm 1,j}\to F_{i\pm 1,j}+\delta F_{i\pm 1,j}\,, \\ F_{i,j\pm 1}\to F_{i,j\pm 1}+\delta F_{i,j\pm 1}\,, \qquad (5) \\ F_{i,j}\to 0\,,$$

where the increases in the nearest-neighboring forces are

$$\delta F_{i\pm 1,j}=\frac{K_1}{2K_1+2K_2+K_L}F_{i,j}=\alpha_1 F_{i,j}\,, \\ \delta F_{i,j\pm 1}=\frac{K_2}{2K_1+2K_2+K_L}F_{i,j}=\alpha_2 F_{i,j}\,. \qquad (6)$$

For simplicity we denote the elastic ratios by α_1 and α_2, respectively. Notice that this relaxation rule is very similar to the well-known BTW model. However, when $K_L>0$ the redistribution of the force is nonconservative. Thus, the well-established spring-block model, used to describe earthquakes, is nonconservative in nature. If $K_1\neq K_2$ $(\alpha_1\neq\alpha_2)$ this model is also anisotropic. We will describe the phase diagram for this case in a forthcoming paper [19]. Furthermore, if $\alpha_1\neq 0$, $\alpha_2=0$, we have a one-dimensional version of the spring-block model. Nakanishi [16,17] studied this model with the relaxation rule for the slipping block i as a function of the excess force F_i-F_{th}, that is, $F_i\to\phi(F_i-F_{th})$. In our model $\phi\equiv 0$ and the one-dimensional nonconservative systems are not critical.

In this paper we restrict ourselves to the isotropic case, $K_1=K_2$ $(\alpha_1=\alpha_2=\alpha)$. The boundary condition of the model is rigid, implying that $F=0$ on the boundary. The time interval between earthquakes is much larger than the actual duration of an earthquake. Thus, the mapping of the spring-block model into a continuous, *nonconservative* cellular automaton modeling earthquakes is described by the following algorithm.

(1) Initialize all sites to a random value between 0 and F_{th}.

(2) If any $F_{i,j}\geq F_{th}$ then redistribute the force on $F_{i,j}$ to its neighbors according to the rule

$$F_{n,n}\to F_{n,n}+\alpha F_{i,j}\,, \\ F_{i,j}\to 0\,, \qquad (7)$$

where $F_{n,n}$ are the strains for the four-nearest neighbors. An earthquake is evolving.

(3) Repeat step 2 until the earthquake is fully evolved.

(4) Locate the block with the largest strain, F_{max}. Add $F_{th}-F_{max}$ to all sites (global perturbation) and return to step 2.

We measure the probability distribution of the size (the total number of relaxations) of the earthquakes, which is proportional to the energy released during an earthquake.

There are several differences between our model and the BTW model.

(i) The strain on the critical site is set to zero when relaxed. (ii) The redistribution of strain to the neighbors is proportional to the strain in the relaxing site [20]. (iii) The relaxation is *not* conservative. It will be conservative *only* when $K_L=0$. If $K_L>0$, the model will be nonconservative. In the context of the spring-block model,

$K_L > 0$; otherwise no driving force is possible. If we assume that all elastic constants are on the same scale ($K_1 \approx K_2 \approx K_L$) then $\alpha \approx 0.20$.

The continuous, nonconservative cellular automaton model exhibits SOC behavior for a wide range of α values. The exponent B depends on α. The dependence is shown in Figs. 2(a) and 2(b). To verify the criticality of the model we study the effect of increasing the system size L. We observe that for α=const the exponent stays the same, while the cutoff in the energy distribution scales with system size. We give an example of this behavior in Fig. 3 which shows the results of simulations with $\alpha = 0.20$ for $L = 15$, 25, 35, and 50, giving $B \approx 0.91$. By examining the scaling of the cutoff in the energy distribution as a function of the system size for $\alpha = 0.20$ we find that the cutoff scales with $L^{2.2}$ [19]. In the original BTW model the cutoff scales with the square of the system size L. This verifies the criticality of the model as well as the absence of any characteristic length scale associated with nonconservation. For the lower values of α ($\alpha \leq 0.10$) it is difficult to pursue this scaling since the large values of B in this region make it very hard to obtain good statistics for the large events.

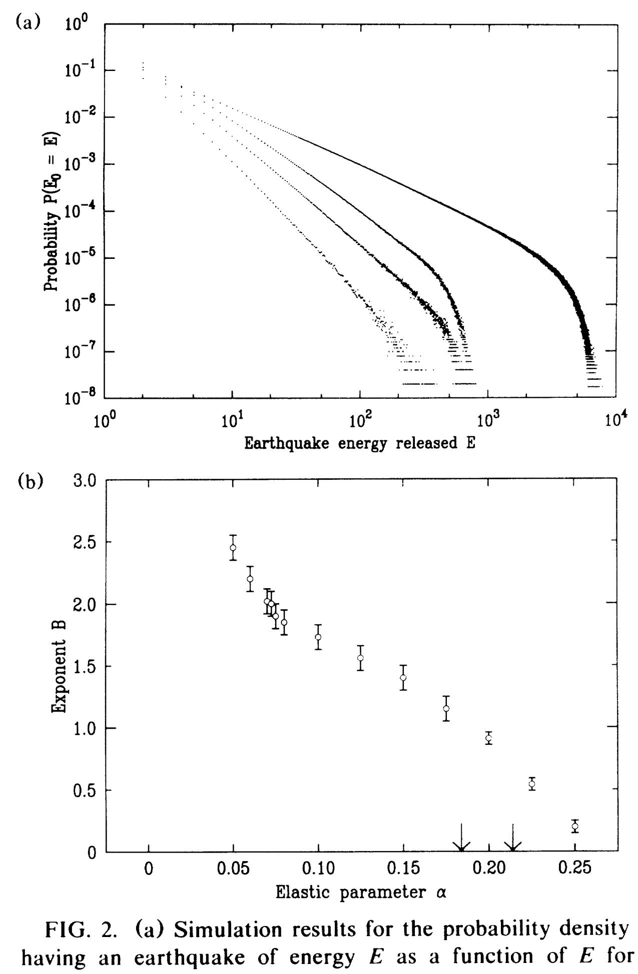

FIG. 2. (a) Simulation results for the probability density of having an earthquake of energy E as a function of E for a 35×35 system. The curves correspond to $\alpha = 0.25$, 0.20, 0.15, and 0.10. The slope of the curves becomes steeper as the α value is decreased. (b) The power-law exponent B as a function of the elastic parameter α. The level of conservation is 4α. Notice the sudden change of the B value around $\alpha = 0.07$. Below $\alpha = 0.05$ there is a transition to exponential decay. The arrows indicate the actual measured B values for earthquakes [7].

It is clear that if $\alpha = 0$ the movement of the blocks will become completely uncorrelated due to the lack of interaction. Therefore, we expect to see a transition to a localized behavior, characterized by a change from a power-law dependence to an exponential decay. This indeed occurs in this model for $\alpha \approx 0.05$. Notice that for this value only 20% of the value at the critical site is redistributed.

There is another transition in the slope of the critical exponent when B reaches the value of 2.0 ($\alpha \approx 0.07$). For this value the variance of the avalanches becomes well defined.

Since other algorithms seem to be very dependent on external noise [5,11] we checked the effect of noise on our model. We introduced noise with zero mean and with a variance up to $0.25F_{\text{th}}$ in each relaxing site for the case of $\alpha = 0.20$. This noise had no effect on the exponent and cutoff.

In conclusion we have shown that a continuous, nonconservative model can have a very robust SOC behavior. This is not in accord with the predictions based on differential equations [2,3]. Previous published models [9–11] neglect the presence of the overlying leaf spring, thus assuming $K_L = 0$. This implies that their models are conservative.

The model has a very wide range of exponents, between 0.22 for the conservative case and 2.5 for the transition point. In addition, if the elastic constants are comparable

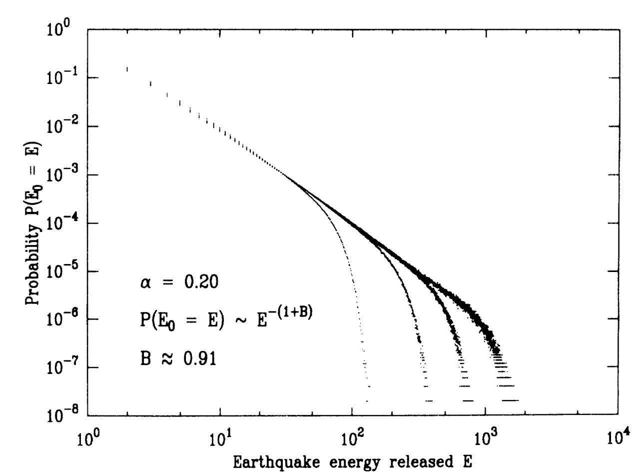

FIG. 3. The probability density of having an earthquake of energy E as a function of E for $\alpha = 0.20$. The different curves refer to different system sizes $L = 15$, 25, 35, and 50. The cutoff in energy distribution scales with $L^{2.2}$.

($K_1 \approx K_2 \approx K_L$) then the characteristic value is $\alpha \approx 0.20$. Therefore, we expect to see the empirically observed B values in the neighborhood of $\alpha \approx 0.20$. The measured range of B values for earthquakes [7] are indicated by the arrows in Fig. 2(b). They are indeed in this region.

Our model is extremely robust even under large noise. On the other hand, the model does not rely on the introduction of noise as is the case in Refs. [5,11]. This is also contradictory to previous models for self-organized criticality based on differential equations [2,3]. Our model includes the basic feature of the spring-block model: the existence of an enormous phase space of *metastable states*. The metastable states lose their stability only if they exceed the threshold value. The models based on driven diffusion equations do not share this feature. We believe this to be the fundamental property modeling earthquakes.

Another interesting feature of our model is the lack of universality. Changing the value of α will alter the exponent but still keep the system critical.

While this model is obviously highly simplified we believe it provides a reasonable resemblance of the actual dynamical process associated with earthquake faults.

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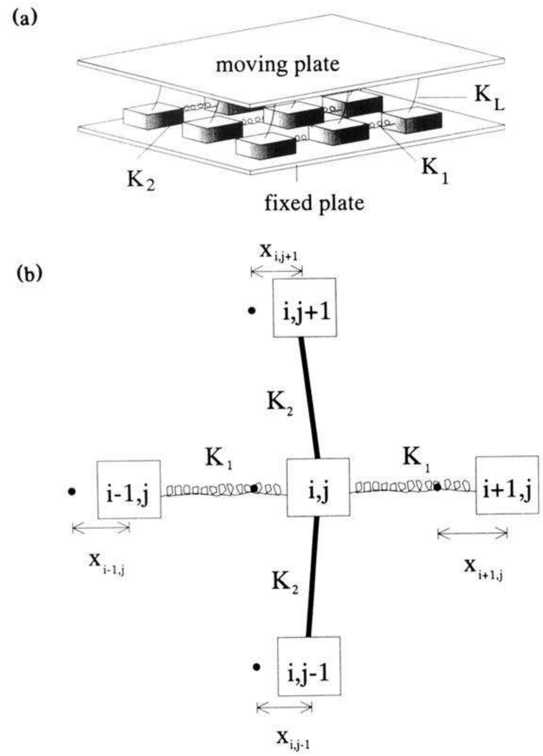

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