Deterministic $1/f$ Noise in Nonconservative Models of Self-Organized Criticality

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Generic, deterministic, nonconservative models displaying self-organized criticality are shown to exhibit $1/f$ noise. The exponent of the power spectrum depends on the level of conservation.

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One of the great mysteries of physics is the $1/f$ noise observed in signals from sources ranging from the light of quasars to the flow of the river Nile and the current flowing through a resistor [1,2]. The noise is called "$1/f$" noise despite the fact that the exponent of the power spectrum is rarely 1, but varies from system to system, and is typically in the range of 0.6 to 1.6. Remarkably, the power law is obeyed over several decades, spanning time scales where one might expect the physics to differ significantly. Using a very simple dynamical model to capture the essence of the underlying mechanism, we argue and demonstrate numerically that $1/f$ noise is a deterministic self-organized critical phenomenon emerging naturally in interactive dissipative dynamical systems with many degrees of freedom.

A few years ago, Bak, Tang, and Wiesenfeld (BTW) [3] discovered that extended dynamical many-body systems self-organize into a critical state. The idea complements the concept of "chaos" wherein simple systems, with a small number of degrees of freedom, can display quite complex behavior. The critical state is characterized by avalanches (activity) with power-law spatial and temporal distribution functions limited only by the size of the system. The original models were driven by white noise, but later it was shown that deterministic models exhibit the same behavior, that is, the criticality is not caused by but, on the contrary, is robust with respect to noise. Thus, a mechanism for establishing coherence over all time scales was provided.

However, the spatiotemporal scaling in the self-organized critical state does not necessarily manifest itself in nontrivial exponents of the power spectrum. Jensen, Christensen, and Fogedby [4] and later Kertész and Kiss [5] showed that the power spectrum of the activity was in fact $1/f^2$, i.e., the spectrum of a random walk. The proper relationship between the joint probability density $P(S=s,\ T=t)$ of having an avalanche of size s and lifetime t and the power spectrum $S(f)$ of linearly superimposed avalanches was worked out by Jensen, Christensen, and Fogedby [4(a)] and later generalized by them [4(b)]. Introducing the *weighted lifetime distribution*

$$\Lambda(t)=\sum_s s^2 P(S=s,\ T=t)\,, \tag{1}$$

and assuming that $\Lambda(t)$ exhibits a scaling behavior

$$\Lambda(t)\sim t^{\mu},\quad 0<t_1\le t\le t_2<\infty\,, \tag{2}$$

and is negligible outside this interval, they showed

$$S(f)\sim\begin{cases}1, & \mu+1<0\,,\\ f^{-(\mu+1)}, & 0<\mu+1<-\alpha_\infty\,,\\ f^{\alpha_\infty}, & \mu+1>-\alpha_\infty\,,\end{cases} \tag{3}$$

for the scaling of the power spectrum in the intermediate frequency regime $1/2\pi t_2 \ll f \ll 1/2\pi t_1$. Here, α_∞ is an exponent characterizing the shape of the avalanches and it can be proven to be ≤ -2. For $f\to 0$ the power spectrum becomes white, since a linearly superimposed signal cannot contain temporal correlations beyond the longest possible lifetime of an avalanche. Thus, in order to get nontrivial exponents, $\mu+1$ must be between 0 and 2, whereas in the BTW model $\mu=3.1$, 2.77, 2.69, and 2.56 in dimension 2, 3, 4, and 5, respectively. Notice that the lower frequency cutoff scales inverse proportionally with the upper temporal cutoff t_2 of the weighted lifetime distribution.

In the BTW model the local dynamical variable was conserved during the relaxations, whereas realistic models of natural phenomena, such as earthquakes, do not typically have any inherent conservation law. For some time it was believed that the introduction of any degree of nonconservation into a BTW-type model would necessarily lead to a finite correlation length [6]. Recently, however, Feder and Feder [7] introduced a two-dimensional, nonconservative model displaying criticality. Shortly afterwards Olami, Feder, and Christensen [8] discovered a class of deterministic models, related to spring-block models of earthquakes, which exhibit self-organized critical behavior with a conservation level down to 20%, and nonuniversal exponents depending on the level of conservation.

The simplicity and robustness of those models suggest that they can be viewed as generic "Ising" models of dissipative many-body systems; hence, we concentrate our effort on those models: A set of dynamical variables $F_{i,j}$, representing the local force (strain) at site (i,j), is defined on a two-dimensional lattice $1\le i,\ j\le L$. The values of $F_{i,j}$ are increased uniformly at an essentially infinitely slow rate until somewhere the force exceeds a critical value $F_{\rm th}$. Then the force on the unstable site, $F_{i,j}$, and the values of the force at its nearest neighbors $F_{\rm nn}$, are updated according to the simple relaxation rule

$$F_{\rm nn}\to F_{\rm nn}+\alpha F_{i,j},\quad F_{i,j}\to 0\,. \tag{4}$$

This initiates an avalanche which lasts for t time units (a *unit time step* is defined as one simultaneous update of the whole lattice) and involves s relaxations. As F keeps increasing, there will be more avalanches triggered by instability on other sites. In the beginning, correlations are short range in time and space. During a long transient period, depending on the size of the system and α, the

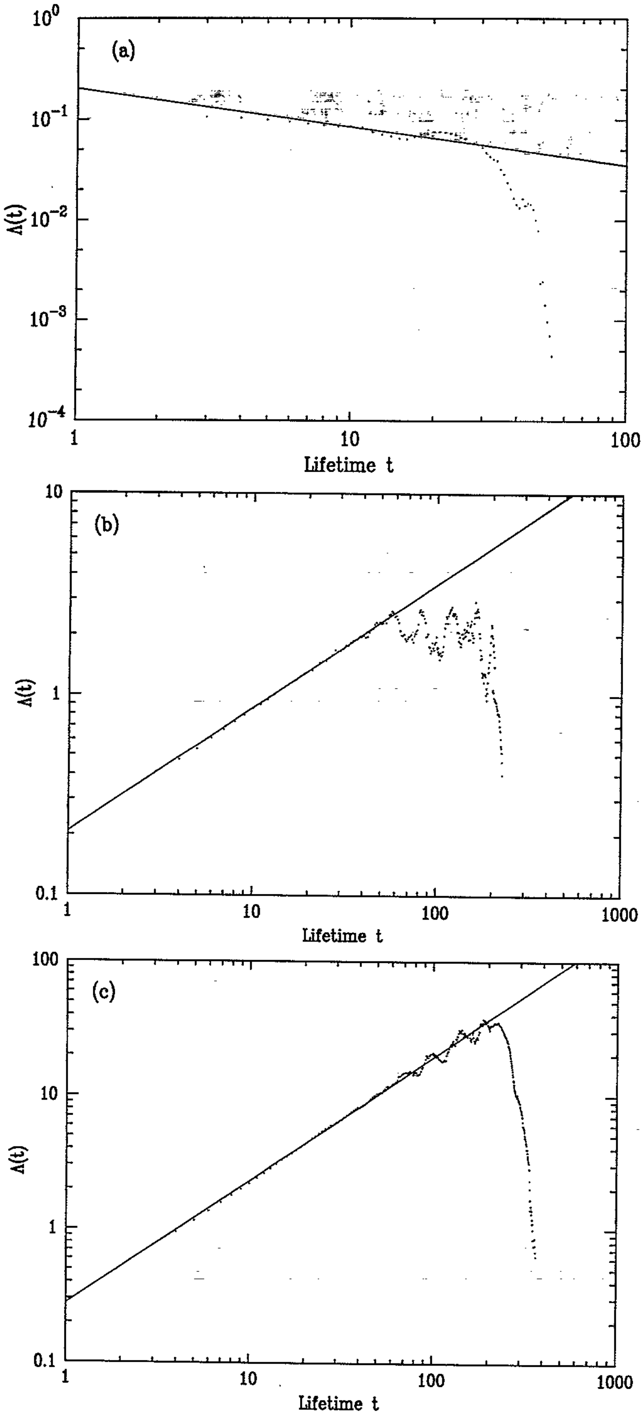

FIG. 1. Distribution of weighted duration of avalanches in a system of size $L=100$. (a) $\alpha=0.10$, (b) $\alpha=0.15$, and (c) $\alpha=0.20$. The measured exponents μ are -0.40, 0.61, and 0.92, respectively.

correlations grow, until saturating at a value limited by and scaling with the size of the system, indicative of the slow self-organizing process leading to the stationary critical state. Notice that the level of conservation is given by 4α.

The difference from the deterministic, continuous version of the original BTW model may seem subtle, but nevertheless has striking consequences. In the BTW model the decrease in the value of the relaxing site was a fixed amount (say 4) and the transfer to the neighbors was independent of the previous state of the system. This rigid way of transferring force did not allow the system to self-organize except for the special case of a conservative system. In the model given above the relaxing amount and the transfer to the neighbors depend on the state of the system. This appears to be a much more general situation. The fact that one has a handle on the exponents, through the level of conservation, suggests that a power spectrum with nonuniversal nontrivial exponents may emerge, as observed in nature.

We now focus on the weighted lifetime distribution for the avalanches defined in Eq. (1) and its relation to the power spectrum of linearly superimposed signals. This is to mimic the total activity in a very large, slowly driven system in which we can neglect the interference between different avalanches.

Figure 1 shows the weighted lifetime distribution $\Lambda(t)$ for various values of α. Indeed, they obey power laws with the exponent μ depending on the dissipation. The values of μ are in the range where nontrivial exponents for the power spectra are expected.

In order to measure directly the power spectrum of linearly superimposed avalanches we generate realizations of the number of relaxations per unit time step $j(\tau)$ by repeating the following procedure: (1) Set $j(\tau)=0$ for all τ. (2) Perturb the system until an avalanche is activated. (3) Choose a starting time τ_0 at random, and increase $j(\tau_0+\tau)$ by the activity (number of relaxations per unit time step), for $\tau=1,\ldots,t$, where t is the lifetime of the avalanche. (4) Go to (2).

The assumption of no interference between different

TABLE I. The measured exponents $\mu+1$ and φ for the weighted lifetime distribution and power spectrum, respectively. The exponents are, within numerical accuracy, consistent with Eq. (3). For $0.0225<\alpha\leq 0.25$ it was not possible to identify any scaling region. For comparison we also list the critical exponents for the original two-dimensional BTW model.

Model	$\mu+1$	φ
$\alpha=0.10$	0.60 ± 0.10	0.50 ± 0.10
$\alpha=0.125$	1.20 ± 0.10	1.18 ± 0.05
$\alpha=0.15$	1.61 ± 0.10	1.56 ± 0.05
$\alpha=0.175$	1.80 ± 0.10	1.76 ± 0.05
$\alpha=0.20$	1.92 ± 0.10	1.80 ± 0.05
$\alpha=0.225$	2.24 ± 0.10	1.92 ± 0.05
2D BTW	3.10 ± 0.10	2.00 ± 0.05

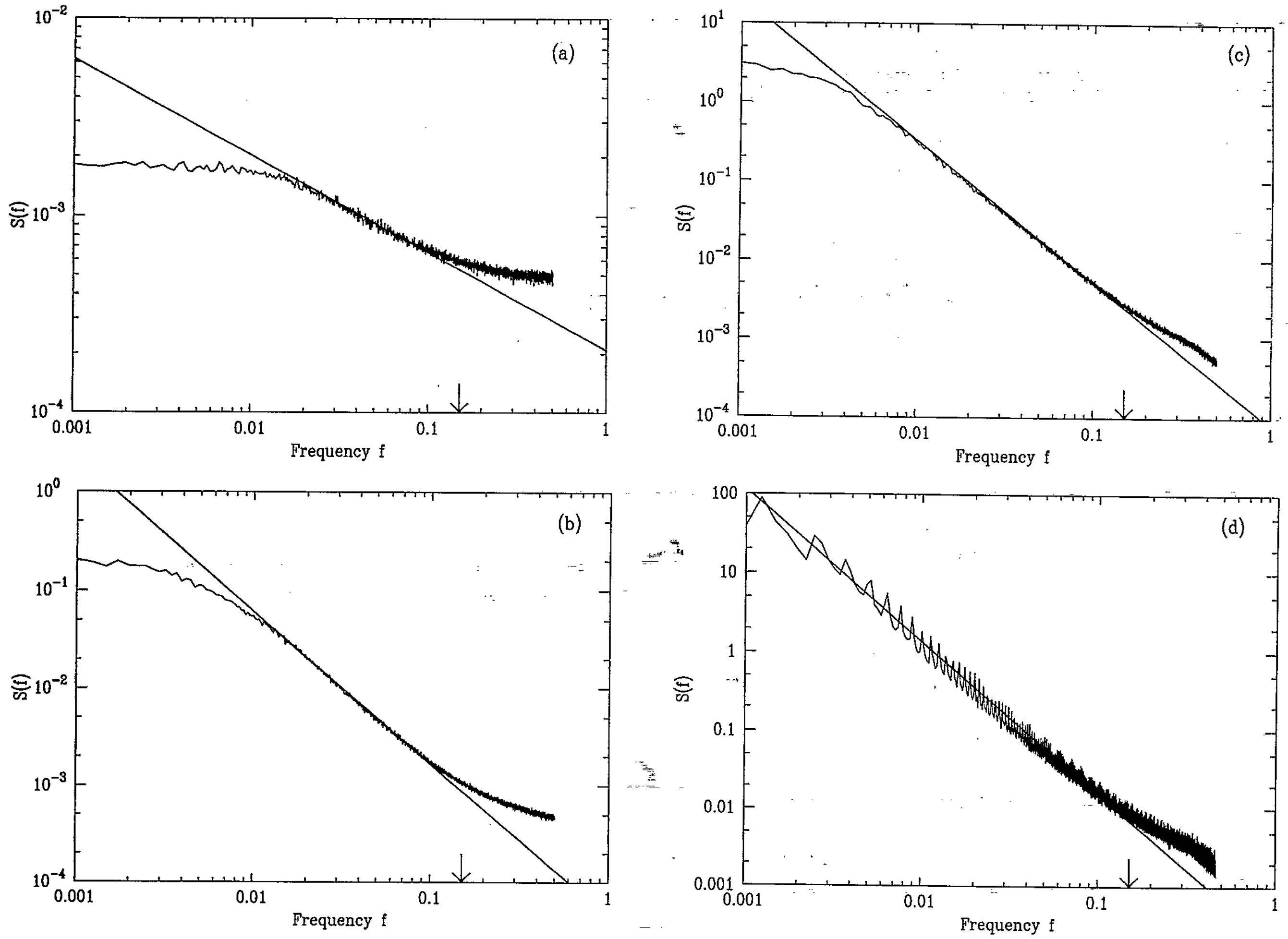

FIG. 2. The power spectrum of randomly superimposed avalanches for the same α values as in Fig. 1. The arrow indicates the upper frequency cutoff $1/2\pi$ for the scaling region. The exponents φ of the power spectrum are (a) $\varphi=0.50$, (b) $\varphi=1.56$, and (c) $\varphi=1.80$. Within numerical accuracy, those values are consistent with the values of μ. (d) The power spectrum of a system of size $L=250$ driven continuously at a rate $p=0.001$ for $\alpha=0.20$. The straight line has a slope of -1.93.

avalanches was essential for the derivation of Eq. (3). Time sequences $j(\tau)$ generated according to the algorithm above, will, of course, fulfill this requirement.

For a given realization $j(\tau)$ we make the Fourier transform $\hat{j}(f)$, and define the power spectrum

$$S(f)=|\hat{j}(f)|^2, \tag{5}$$

which is a strongly fluctuating function of the frequency f. We average over many different realizations (up to 1000 realizations) to reduce the variance in the power spectrum, which for one single realization is a 100% standard deviation. Figures 2(a)–2(c) display the resulting power spectra measured by randomly superimposing the avalanches in a system of size $L=100$. We recall that the frequency region under consideration is $1/2\pi t_2 \ll f \ll 1/2\pi$, since $t_1=1$ by definition. The slopes of the straight lines are indeed in agreement with the predicted values obtained from substituting the values from Fig. 1 into Eq. (3). Table I lists all our simulation results. The exponents are roughly in the regime observed experimentally. Note the change in the sign of the slope μ is related to a change of the power-law exponent from values less than unity to values greater than unity.

Alternatively, the time sequence $j(\tau)$ can be generated by a direct measurement of the activity in a slowly driven system of considerable size. Figure 2(d) presents the power spectrum of a system of size $L=250$ driven with a finite rate $p=0.001$ for $\alpha=0.20$. The long-term correlations ignored by the random superposition method do not affect the shorter time scales of interest here. Indeed, the power spectrum seems to be in even better agreement with the exponent extracted from the exponent μ of the weighted lifetime distribution.

Finally, in order to verify that the interval in which the power spectrum displays $1/f$ noise scales with system size, we generated the power spectrum for various system

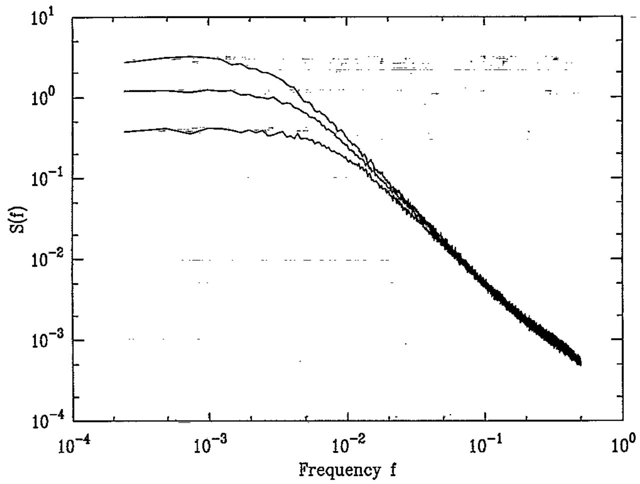

FIG. 3. The power spectrum for systems with $\alpha = 0.20$ but different system sizes, $L = 45, 70, 100$. The lower frequency cutoff scales with system size, while the upper frequency cutoff is a constant. The exponent of the power spectrum seems to change slightly with system size.

sizes, $L = 45, 70, 100$, for $\alpha = 0.20$. The power spectrum is shown in Fig. 3. We observe that the lower frequency cutoff scales with system size L. This is a unique fingerprint of a many-body phenomenon. Hence, with this characteristic in mind, we urge that experiments be performed on systems of varying size in order to check the assertion that the $1/f$ noise is a critical dynamical many-body effect.

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[1] W. H. Press, Comments Astrophys. **7**, 103 (1978).
[2] J. Feder, *Fractals* (Plenum, New York, 1988).
[3] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988); C. Tang and P. Bak, Phys. Rev. Lett. **60**, 2347 (1988); P. Bak, K. Chen, and M. Creutz, Nature (London) **342**, 780 (1989); P. Bak, K. Chen, and C. Tang, Phys. Lett. A **147**, 297 (1990).
[4] (a) H. J. Jensen, K. Christensen, and H. C. Fogedby, Phys. Rev. B **40**, 7425 (1989); (b) K. Christensen, H. C. Fogedby, and H. J. Jensen, J. Stat. Phys. **63**, 653 (1991); (c) K. Christensen, in *Spontaneous Formation of Space Time Structure and Criticality*, edited by T. Riste and D. Sherrington (Academic, New York, 1991), p. 33.
[5] J. Kertész and L. B. Kiss, J. Phys. A **23**, L433 (1990).
[6] T. Hwa and M. Kardar, Phys. Rev. Lett. **62**, 1813 (1989); G. Grinstein, D.-H. Lee, and S. Sachdev, Phys. Rev. Lett. **64**, 1927 (1990); S. S. Manna, L. B. Kiss, and J. Kertész, J. Stat. Phys. **61**, 923 (1990).
[7] H. J. S. Feder and J. Feder, Phys. Rev. Lett. **66**, 2669 (1991); Phys. Rev. Lett. **67**, 283(E) (1991).
[8] Z. Olami, H. J. S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992); K. Christensen and Z. Olami, J. Geophys. Res. (to be published); "Scaling, Phase Transitions, and Nonuniversality in a Self-Organized Critical Cellular Automaton Model" (to be published).