Self-Organized Critical Forest-Fire Model: Mean-Field Theory and Simulation Results in 1 to 6 Dimensions

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We argue that the critical forest-fire model (FFM) introduced by Drossel and Schwabl [Phys. Rev. Lett. **69**, 1629 (1992)] is a critical branching process in the mean-field approximation where the number s of trees burned in a forest fire is power-law distributed with exponent $\tau_s = 5/2$. The mean-field model of the FFM in finite dimension is the percolation model and, as in percolation, the upper critical dimension is 6. Simulations show that the FFM becomes increasingly percolationlike with increasing dimension d and is, within error bars, fully consistent with the percolation results when $d \gtrsim 3$.

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Recently Drossel and Schwabl [1] proposed a new, critical version of the forest-fire model originally introduced by Bak, Chen, and Tang [2]. The model is defined on a d-dimensional hypercubic lattice of linear size L. A lattice site is either *empty*, a *tree*, or a *burning tree*. At every time step the configuration evolves according to the following rules: (1) A tree grows in an empty site with probability p. (2) A site with a burning tree becomes an empty site. (3) A tree becomes a burning tree if one or more of its neighbors is a burning tree. (4) A tree without burning neighbors catches fire spontaneously with probability f.

The lightning probability f was introduced in Ref. [1], where it was shown that this addition makes the forest-fire model critical in the limit $f/p \to 0$, provided the duration of individual forest fires is much smaller than the time scale of growing trees, i.e., $f \ll p \ll 1/T(\langle s \rangle)$, where $T(\langle s \rangle)$ is the time during which a forest of the average size $\langle s \rangle$ is burned down. Furthermore, scaling laws were derived leading to the conclusion that the power-law exponent τ_s characterizing the size distribution of forest fires equals 2 in any dimension d. However, the derivation relies on the assumption that the cutoff in the cluster size distribution s_ξ and the average size of forest fires $\langle s \rangle$ have the same scaling exponent. This is not true in general: In percolation theory, for example, the cutoff diverges with an exponent $1/\sigma$ while the average cluster size diverges with an exponent $\gamma = (3-\tau_s)/\sigma$; see, e.g., Ref. [3]. The assumption $1/\sigma = \gamma$ results in $\tau_s = 2$, but this is correct only in 1D for the percolation problem.

We show below that in a mean-field description the size of forest fires is power-law distributed with exponent $\tau_s = 5/2$. One would expect the mean-field description to be exact either above an upper critical dimension d_u or in the limit $d \to \infty$, and one may ask how $\tau_s(d)$ approaches its mean-field value as d increases.

First, we give a mean-field version, or, more precisely, a *random neighbor* version, of the forest-fire model: We disregard the lattice geometry and consider instead an ensemble of N sites on which trees may grow. The system evolves according to the dynamical rules defined above, but at each time step every site with a burning tree is assigned z "neighbor" sites to which the fire will spread to the extent that there are trees on these sites. This neighbor relationship is oriented: fire may spread only one way through it. Neighbor sites are chosen at random from the ensemble and rechosen anew at every time step. Thus the name "random neighbor" model. The repeated rechoosing of random neighbors may sound complicated, but is actually a decisive simplification when it comes to calculations. The parameter z is called the *coordination number* and is identified with $2d-1$ when the random neighbor model is used as an approximation to the system on a hypercubical lattice. Let $\rho_e(\tau)$, $\rho_t(\tau)$, and $\rho_f(\tau)$ denote the densities of empty sites, trees, and burning trees at time τ. The mean-field equations for the forest-fire model are the rate equations for these densities supplemented with the normalization constraint. For the small values of the lightning rate f we will consider, we may assume—and later find correct—that the number of burning trees at any given time is small compared to the total number of trees. "Small" here means $N\rho_f \lesssim \sqrt{N\rho_t}$. With this assumption, no tree is ignited by more than one burning tree, and the rate equations read

$$\begin{aligned}
\rho_e(\tau+1) &= (1-p)\rho_e(\tau) + \rho_f(\tau)\,, \\
\rho_t(\tau+1) &= [1-f-z\rho_f(\tau)]\rho_t(\tau) + p\rho_e(\tau)\,, \\
\rho_f(\tau+1) &= [f+z\rho_f(\tau)]\rho_t(\tau)\,, \\
\rho_e(\tau) &+ \rho_t(\tau) + \rho_f(\tau) = 1\,.
\end{aligned} \tag{1}$$

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The time evolution of these equations has an attractive fixed point. Its existence and stability may be understood as follows: With many trees in the forest, fire will propagate easily, and more trees will disappear than are grown. With few trees in the forest, fire will propagate with difficulty, die out fast, and more trees will grow than disappear. While oscillations between these two situations certainly occur locally in finite dimensions, the mean-field equations have an attractive fixed point, and consequently the system will reach a stationary state.

Introducing

$$k=f(1+p)/p\,, \tag{2}$$

we solve the stationary version of Eq. (1) with Taylor series in k, and find for the density of trees

$$\rho_t=\frac{z+1+k\pm\sqrt{(z-1)^2+2(z+1)k+k^2}}{2z}$$
$$=\frac{1}{z}-\frac{1}{z^2-z}k+O(k^2)\,. \tag{3}$$

Only the solution with the minus sign is meaningful, the other giving $\rho_t>1$.

Let us describe the forest fire as a random branching process: One burning tree can ignite from 0 to z other trees, depending on how many of its z neighbor sites are occupied by trees. By assumption, the fire is so sparse at any time that no tree is ignited by more than one burning tree. Consequently, a space-time map of a forest fire has the topology of a tree, each node representing a burning tree, and branches from such a node representing the spreading of the fire to neighbor sites. Since the fire can spread from one burning tree to any number b of trees between 0 and z, the average number of trees ignited by one burning tree is

$$\langle b\rangle=\sum_{b=0}^{z}b\binom{z}{b}\rho_t^b(1-\rho_t)^{z-b}$$
$$=1-\frac{1}{z-1}k+zO(k^2)\,, \tag{4}$$

where, in the first identity, we have used that a randomly chosen site contains a tree with probability ρ_t. Since k is proportional to f, in the limit of $f\to 0$ $\langle b\rangle\to 1$. In this limit the mean-field theory of the forest-fire model will be nothing but a critical branching process—*critical* because $\langle b\rangle=1$ means a burning tree on the average ignites exactly one tree. Thus the fire continues forever, on the average. If $k\neq 0$ then $\langle b\rangle<1$ and the system is subcritical, and a fire will die out in a finite time. For random branching processes, the number $N(s)$ of processes with s nodes is known asymptotically for large values of s:

$$N(s)\propto s^{1-\tau_s}\exp(-s/s_\xi)\,. \tag{5}$$

Here $\tau_s=5/2$, and the cutoff in cluster size distribution s_ξ diverges when $\langle b\rangle\to 1$; see, e.g., Ref. [4]. Simulations of the random neighbor model are consistent with our analytical results and the assumption, $N\rho_f\lesssim\sqrt{N\rho_t}$, they are based on.

Now let the geometry of the model (i.e., the neighbor relations) be given by an underlying lattice of finite dimension d. This will change the process in two ways: (1) A fire will self-interact, resulting in different critical exponents, and (2) the dynamical process might induce correlations between sites. In an attempt to keep some features of the mean-field theory, we assume that sites are not correlated, and that the lattice has an average density of trees ρ_t. Then we have a *percolation* problem: A fire will burn exactly the cluster of trees it was started in, and the known cluster-size distribution of the percolation problem is our mean-field estimate for the size distribution for forest fires in finite dimensions. In particular, the known exponents of percolation theory are our mean-field estimates for exponents in the forest-fire model.

The density of trees can be derived self-consistently using the knowledge of the exponents from percolation theory: In a statistically stationary state the rate of flow into the system (the rate of growth) equals the rate of flow out of the system (the rate of burning), that is, if $\langle s\rangle$ denotes the average size of a forest fire initiated by a lightning, then

$$p\rho_e L^d=\langle s\rangle f\rho_t L^d\,, \tag{6}$$

and using $\rho_e=1-\rho_t$ we have

$$\langle s\rangle=\frac{p}{f}\frac{1-\rho_t}{\rho_t}\propto|p_c-\rho_t|^{-\gamma}\,, \tag{7}$$

since the average cluster in percolation theory diverges with the exponent γ. This gives an estimate of the density of trees as a function of p and f. However, we can also estimate γ using the measured value of ρ_t. For this estimate to be consistent, the correlation length in the percolation problem should be much smaller than the lattice size so that fluctuations in ρ_t are negligible. Since the fractal dimension of clusters in the percolation problem is smaller than the embedding dimension when $d\geq 2$ [5], we expect that density fluctuations indeed will be unimportant.

This approximate mapping of the forest-fire problem into percolation theory also suggests that the forest-fire problem has an upper critical dimension d_u with value 6, like the percolation problem. Note that our first description of the forest-fire model as a random branching process and its second description in terms of percolation theory are identical when $d\geq d_u=6$, because percolation theory is exactly described by mean-field theory above its upper critical dimension. The mean-field theory of the percolation problem is identical to the one we devised for the forest-fire model when z is identified with $2d-1$.

It can easily be shown for random branching processes which do not self-interact that their extent s scales with their duration t as $s\propto t^2$, on the average [4]. If a random branching process propagates in a finite dimensional space, the radius of the cluster of sites it affects will obviously scale as $r \propto t^{1/2}$ since the process may be thought of as a random walk with branching and death. Consequently, if a random branching process can take place in a finite dimensional space *and* not self-interact, we have $s \propto r^4$. Thus we expect to find $s \propto r^4$ when the self-interaction becomes unimportant. This last relationship is a well known result for percolation clusters in dimensions $d \gtrsim 6$, where loops (self-interactions) become relatively unimportant. This indicates, that a *branching*, self-interacting process has an upper critical dimension $d_u \gtrsim 6$.

Similar arguments can be made for the Bak-Tang-Wiesenfeld sandpile model [6]: Since avalanches (the branching process) self-interact, the critical dimension should be larger than or equal to 6. It is possible to formulate a random neighbor process of the sandpile model which results in a power-law exponent of $\tau_s = 5/2$ [7]. The same exponent appears in several models: The sandpile on the Bethe lattice [8], the sandpile model with local (stochastic) nonconservative relaxation rules that globally conserves the dynamical variable [9], and in a globally coupled (conservative) spring-block model [10].

To check the validity of our arguments and results, we have simulated the forest-fire model in dimensions 1 to 6. Figure 1 shows our results for the size distribution in various dimensions. We find that the size of fires is power-law distributed with an exponent that does indeed change with dimension, as shown in Table I [11].

Table I also shows the value of the exponent $\tau_s^{(\mathrm{perc})}(d)$ characterizing the cluster-size distribution in percolation at the percolation threshold. The exponent $\tau_s(d)$ for forest fires barely differs from that of percolation. To the extent it does differ, it seems to converge to it with increasing d, the results above two dimensions being fully consistent with the percolation results within one standard deviation. A result by P. Grassberger also indicates that the exponents do differ for $d=2$. He did a large-scale simulation with $L=8192$ and $p/f=4000$ and found $\tau_s = 2.15 \pm 0.02$ which differs from $\tau_s^{(\mathrm{perc})} = 187/91$ by 4-5 standard deviations [5].

This is also supported by the estimates we obtain for the γ exponent using Eq. (7): Since the proportionality factor is unknown, we measured ρ_t in four systems with different values of p/f in order to determine γ. From the theory of percolation we know that the proportionality factor depends only very weakly on the distance from p_c. Note that p_c is to be interpreted as the limit of ρ_t when $f/p \to 0$. In 3D we find $\gamma = 1.74(10)$ [with $p_c = 0.222(2)$] which is to be compared with the percolation result $\gamma^{(\mathrm{perc})} = 1.80$ in 3D. In 2D, however, we find $\gamma = 2.03(10)$ [with $p_c = 0.409(2)$, see also [5]] which is far from the percolation result of $\gamma^{(\mathrm{perc})} = 43/18$. Again, the result in 2D is inconsistent with percolation theory, but in 3D (and higher dimensions) the results are indistinguishable from that of percolation theory.

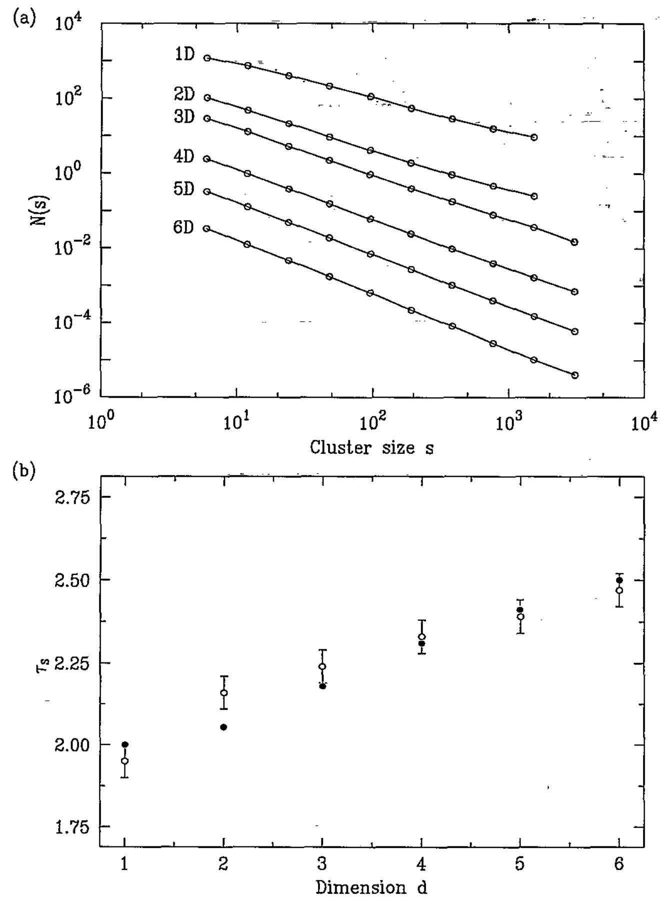

FIG. 1. (a) The size distribution of fires $N(s)$ for different dimensions. We only display the region used to determine the power-law exponent. The data are shifted along the y axis in order to separate the graphs. $p/f = 20000$ in 1D while $p/f = 133$ in all the other simulations. 1D with $L = 10^6$, 2D with $L = 1000$, 3D with $L = 200$, 4D with $L = 50$, 5D with $L = 25$, and 6D with $L = 10$. (b) The power-law exponents τ_s in the forest-fire model (open symbols) and the percolation model (solid symbols).

Additional support is offered by the measurements of the density of trees given in Table II for $d = 1$, 2, 3, 4, 5, and 6. This density may be compared with the results one would expect in a random neighbor approximation

TABLE I. First row: dimension d of space. Second row: measured critical exponent $\tau_s(d)$ for size distribution of forest fires. The analytical result for the 1D forest-fire model is $\tau_s = 2$ [11]. Third row: exponent $\tau_s^{(\mathrm{perc})}(d)$ for the cluster size distribution at percolation threshold ([3], p. 52). For $d \gtrsim 3$ the error bars on these exponents is on the last digit, while the results in $d=1$ and 2 are exact.

d	1	2	3	4	5	6
$\tau_s(d)$	1.95(5)	2.16(5)	2.24(5)	2.33(5)	2.39(5)	2.47(5)
$\tau_s^{(\mathrm{perc})}(d)$	2	187/91	2.18	2.31	2.41	5/2

TABLE II. First row: dimension d of lattice. Second row: density ρ_t of trees in forest-fire model. Third row: random neighbor result for density of trees. Fourth row: density of occupied sites at site-percolation threshold ([3], p. 17).

d	1	2	3	4	5	6
ρ_t	0.90(1)	0.38(1)	0.22(1)	0.15(1)	0.12(1)	0.10(1)
$1/(2d-1)$	1	0.33	0.20	0.14	0.11	0.091
p_c(site)	1	0.5927	0.3116	0.197	0.141	0.107

$\rho_t = 1/z$ with $z = 2d-1$ which is also given in the table. The coordination number z is identified with $2d-1$ because a fire cannot propagate backwards to a site it came from in the previous time step when p, the probability per time step of growing a new tree, is small as it is here.

In conclusion, we have demonstrated that the sizes of forest fires are power-law distributed with an exponent $\tau_s(d)$ which depends on the dimension d of space for $d \le 6$. This dependence on d is similar but not equal to that previously found in the exponent for the critical cluster size distribution in percolation theory. The two exponents converge with increasing dimension and are indistinguishable for $d \geq 3$ because of the finite numerical precision of our results. We conjecture that the upper critical dimension for the forest-fire model $d_u = 6$, as it is for percolation.

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[11] The result in 1D is for a system of size $L = 10^6$ with $p/f = 20\,000$, but the value of the power-law exponent τ_s was still increasing, though very slowly, when decreasing the ratio f/p. Thus the value $\tau_s = 1.95(5)$ obtained with $p/f = 20\,000$ may differ a little from the true critical exponent found in the limit $p/f \to \infty$. We expect this systematic error to be negligible in comparison with the 2.5% stochastic error on our result, and cannot distinguish our result from the value 2 obtained in percolation theory. (The 1D percolation result is special in the sense that at the percolation threshold $p_c = 1$ there are no clusters left.) Indeed, it has been shown analytically that $\tau_s = 2$ in the 1D forest-fire model [B. Drossel, S. Clar, and F. Schwabl (to be published)].