Comment on "Self-Organized Criticality in the Olami-Feder-Christensen Model"

Recently, de Carvalho and Prado [1] studied the criticality of the Olami-Feder-Christensen (OFC) model [2]. Based on measurements of a branching ratio, they concluded that the 2D OFC model on a square lattice is critical only in the conservative limit $\alpha \to 0.25$. In this Comment, we reconsider some of the results obtained in Ref. [1]. We show that a well-known relationship between the branching ratio and average number of nodes for *uncorrelated* branching processes is also valid in the *correlated* case. Using this exact relation to analyze the measured branching ratio in the OFC model, we find the data to be consistent with a critical model, even when nonconservative, contrary to the conclusion reached by de Carvalho and Prado.

A branching process is defined as a sequence of random variables $\{X_j\}_{j=0}^{\infty}$, in which X_j represents the number of nodes in the jth generation of a tree with $P(X_0 = 1) = 1$. The probability that a node has b branches is denoted by p_b, where $\sum_{b=0}^{\infty} p_b = 1$. For an *uncorrelated* branching process, the nodes have identical p_b's. The branching ratio $\sigma = \sum_b b p_b$ is the average number of branches per node and will determine whether the process is subcritical $\sigma < 1$, critical $\sigma = 1$, or supercritical $\sigma > 1$. The average number of nodes S in an ensemble of trees and the branching ratio σ are related by (see, e.g., Ref. [3])

$$S = \begin{cases} \frac{1}{1-\sigma} & \text{for } \sigma < 1 \\ \infty & \text{for } \sigma \geq 1 \,. \end{cases} \tag{1}$$

Consider the avalanches in the OFC model on a square lattice of linear size L and the level of conservation α a correlated branching process. The toppling sites are equivalent to the nodes in branching process. For each node, we can measure the number b of new sites triggered to topple, and the branching ratio is the average number of new active sites produced by one toppling. If $N_n(b)$ denotes the number of sites causing b topplings, then each (finite) avalanche n has an associated branching ratio

$$\sigma_n = \sum_b b \frac{N_n(b)}{\sum_b N_n(b)} = \frac{s_n - 1}{s_n}, \tag{2}$$

where s_n is the (final) size of the avalanche. When taking an ensemble average over many avalanches, each avalanche has a weight proportional to its size, implying

$$\sigma(\alpha, L) = \frac{\sum_n \sigma_n s_n}{\sum_n s_n} = 1 - \frac{1}{S(\alpha, L)}, \tag{3}$$

where $S(\alpha, L)$ is the average avalanche size. This is the generalization of the mean-field result Eq. (1) for $\sigma < 1$. Note that Eq. (3) is not fulfilled for all the data presented in Ref. [1]. There are two possible scenarios: Either $S(\alpha, L)$ keeps increasing with system size L or $S(\alpha, L)$ approaches a finite constant as $L \to \infty$. Only if $S(\alpha, L) \to$ const will $\lim_{L\to\infty} \sigma(\alpha, L)$ be less that 1.

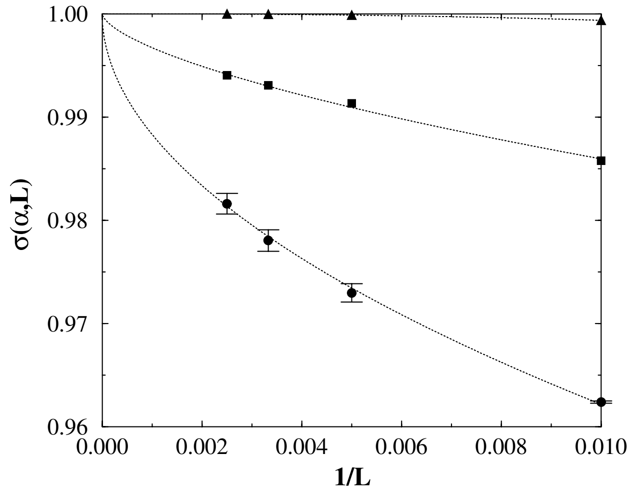

FIG. 1. The branching ratio $\sigma(\alpha, L)$ in the OFC model for $L = 100$, 200, 300, and 400. The displayed points correspond to $\alpha = 0.23$ (circles), $\alpha = 0.24$ (squares), and $\alpha = 0.25$ (triangles). The dashed lines are of the form $\sigma(\alpha, L) = 1 - 1/A(\alpha)L^{\eta(\alpha)}$. An estimate of the error on the measured branching ratios is displayed when it exceeds the size of the symbol.

In Ref. [1] the branching ratio $\sigma(\alpha, L)$ was measured for various (α, L) values. The authors concluded that the branching ratio tends to a value smaller than 1 in the nonconservative case ($\alpha < 1/4$) in the limit $L \to \infty$. By using the same model parameters as Ref. [1], we have repeated the measures for $\sigma(\alpha, L)$, see Fig. 1. Based on these data, one cannot numerically rule out the possibility $\sigma(\alpha, L) \to 1$ and hence that the OFC model remains critical in the dissipative case contrary to the conclusion advocated in Ref. [1].

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Kim Christensen
Blackett Laboratory, Imperial College
Prince Consort Road
London SW7 2BW, United Kingdom

Dominic Hamon, Henrik J. Jensen, and Stefano Lise
Department of Mathematics, Imperial College
180 Queen's Gate
London SW7 2BZ, United Kingdom

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