Self-Organization in Models
of
Sandpiles, Earthquakes, and Flashing Fireflies.

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# Contents

1 Introduction .................................................. 5

2 The branching process ........................................... 11
   2.1 The random neighbor model ................................. 12
      2.1.1 Definition of the random neighbor model .......... 12
      2.1.2 The statistically stationary state ................. 13
      2.1.3 The avalanche size distribution ................. 15
      2.1.4 The interpretation of the parameter $\alpha$ ....... 17
   2.2 Models within the same universality class .............. 18
   2.3 Mean-field theory of the forest-fire model ............. 19
   2.4 Summary .................................................. 21

3 Models with a spatial structure ............................... 23
   3.1 Definition of the model .................................. 23
      3.1.1 The attractor of the system .......................... 24
   3.2 The implications of the spatial structure ............... 24
   3.3 Scaling arguments ........................................ 25
      3.3.1 Scaling of the average avalanche size .............. 25
      3.3.2 Finite-size scaling and scaling relations ......... 26
      3.3.3 Scaling exponent determined by compactness and isotropy ..... 28
      3.3.4 Scaling relation induced by central site multiple topplings . 30
      3.3.5 The dynamical exponent $\gamma_{tr}$ .................. 30
      3.3.6 The interpretation of the fractal boundary ......... 31
      3.3.7 Conjecture of the values of the critical exponents ... 33
   3.4 Summary .................................................. 36

4 Nonconservative models ....................................... 37
   4.1 A laymans geophysics .................................... 37
   4.2 The earthquake mechanism ................................ 38
CONTENTS

4.2.1 Measurements of earthquakes ........................................... 40
4.3 Self-similarities in earthquakes ........................................... 41
4.3.1 The energy-frequency relation of earthquake occurrence ....... 41
4.3.2 Spatiotemporal behavior of earthquakes ............................ 42
4.4 Earthquake modeling .......................................................... 42
4.4.1 A two-dimensional spring-block model ............................... 43
4.4.2 Coupled oscillators ......................................................... 47
4.4.3 Global versus local drive ................................................. 50
4.5 Summary ............................................................................ 52

5 The spring-block model ............................................................ 55
5.1 Self-organization ................................................................. 55
5.2 Distribution function, scaling, and critical indices .................. 56
5.2.1 Finite-size scaling ............................................................... 58
5.2.2 Boundary conditions ......................................................... 59
5.3 Complex spatiotemporal behavior .......................................... 62
5.3.1 Multifractality in the temporal set of earthquake ............... 67
5.4 Summary ............................................................................ 71

6 1/f noise .................................................................................. 73
6.1 The weighted lifetime distribution .......................................... 73
6.2 Measurements in the spring-block model ............................... 74
6.2.1 The weighted lifetime distribution ...................................... 74
6.2.2 The power spectrum .......................................................... 75
6.2.3 Scaling with system size ..................................................... 77
6.3 Summary ............................................................................ 78

7 Acknowledgments .................................................................... 79

A .................................................................................................. 81

B .................................................................................................. 87
B.1 J. Geophys. Res. 97, 8729-8735 (1992) ................................. 87

C .................................................................................................. 95

D .................................................................................................. 107
CONTENTS

E 113
  E.1 Phys. Rev. A 46, R1720-R1723 (1992) . . . . . . . . . . . . . . . . . . . . . . . . 113

F  Dansk resumé 119
Chapter 1

Introduction

A proper understanding of the physical origin of the spatial fractal structures and the temporal fractal behavior observed in physical systems is still lacking.

Fractal structures appear in a variety of physical systems: from the patterns in snowflakes and cauliflowers to the distribution of galaxies [1, 2]. Turbulence is another example of a physical phenomenon displaying fractality. The energy is not dissipated uniformly in space but intermittently through cascades of all length scales.

Another extended system displaying fractal properties is the seismic system. The fault system displays scale-invariant behavior; measurements of the number of faults as a function of length is expressed as a power law. The energy-frequency relation of earthquake occurrence, which is related to the Gutenberg-Richter law [3], is a power-law distribution, see Figure 1.1(a). Also, the geometrical distribution of epicenters (the projection of the nucleation center of earthquakes to the surface of the earth) shows a fractal behavior. Furthermore, the occurrence of earthquakes displays a temporal fractal behavior. As early as in 1894 F. Omori discovered that the number of aftershocks decrease like $1/t$, $t$ being the elapsed time after the main event [4], see Figure 1.1(b). Apart from these short-range temporal correlations, long-range temporal correlations exist between large earthquakes. Contrary to common understanding, the occurrence of large earthquakes is neither periodic nor random: Large earthquakes tend to occur in temporal clusters [5].

In fact, the seismic system has a lot in common with the turbulent system [6]: (1) both are extended dynamical systems with many interacting degrees of freedom, (2) the systems are driven in the long wavelength limit, i.e., energy is fed into the systems uniformly, (3) the energy is dissipated through cascades of all length scales, (4) the dissipation of energy displays temporal intermittency, and (5) both systems display additional spatiotemporal fractal behavior.
CHAPTER 1. INTRODUCTION

Figure 1.1: (a) The number of earthquakes with an energy release greater than $E$. The data, collected by A. C. Johnston and S. J. Nava, are from the New Madrid zone during the period 1974 – 1983 [7]. (b) The distribution of aftershocks of magnitude 3 or greater of the Loma Prieta earthquake, which occurred south of San Francisco on October 17, 1989. The number of aftershocks decreases like $1/t^{1.2}$ [8].

Temporal fractal behavior, known as $1/f$ noise, is seen in many different systems. The phenomenon is observed in signals ranging from the light of quasars to the flow of the river Nile and the current flowing through a resistor [2, 9, 10]. The noise is called “$1/f$” despite the fact that the exponent of the power spectrum rarely is 1, but varies from system to system, typically in the range of 0.6 to 1.6. Remarkably, the power law is maintained over several decades spanning time scales where one otherwise could expect the physics to differ significantly.

The ubiquity of spatiotemporal fractal behavior cannot be accidental. It is most likely that there exists an underlying pattern which we have not yet been able to identify. The present thesis tells the tale of the search for the scheme of things:

A few years ago P. Bak, C. Tang, and K. Wiesenfeld (BTW) suggested that the frequent occurrence of $1/f$ noise and fractal structures is the generic temporal and spatial characteristic of a dynamical critical state into which dynamical systems with many spatial degrees of freedom evolve naturally [11]. Unlike phase transitions in an equilibrium system, a driven dissipative dynamical many-body-system reaches the critical state without the need to adjust the system parameters, i.e., the critical state studied by BTW is an attractor of the dynamics. Therefore, the critical state is usually described as being self-organized, and the phenomena of self-organized criticality may very well provide a connection between the occurrence of $1/f$ noise.
and fractal structures, as well as being the physical origin of these two phenomena.

This idea assumes increasing importance in the new era of physics, where the focus is on “developing complexity out of simplicity” in contrast with the attempt “to reduce complexity to simplicity” to use the words of P. W. Anderson [12]. Fractal growth phenomena such as diffusion-limited aggregation (in which particles perform a random walk until they reach the growing cluster where they come to rest) is an example in which simple local rules lead to a great variety of physical phenomena including scale-invariance [2, 13]. And indeed the relationship between the simple underlying rules governing the dynamics of extended physical systems and the emerging complex structures is an intriguing problem [14].

Often a sandpile is used as a paradigm of an extended many-body system displaying self-organized criticality. As an example: Take a square table and a large bucket of sand. We begin sprinkling grains of sand on the table, one grain at a time. We drop the grain on a randomly chosen place on the table and repeat the act when all motion has terminated. In the beginning the grains just fall on the table in no particular pattern. But subsequently, while continuing to add new grains of sand, we notice the formation of small local avalanches. The mechanism of the local avalanches is to decrease the local slopes whenever they become too steep. Perturbing the system the small sandpiles, being provoked by avalanches, create still greater sandpiles and eventually we end up with only one big sandpile. At some point (the transient time) this pile ceases to grow. The (global) average slope has reached a steady state corresponding to the angle of repose which the sandpile cannot exceed no matter how much sand we add. The pile has reached a statistically stationary state and additional grains of sand will ultimately fall off the pile.

The avalanches are clearly necessary in order to relax the sandpile but we notice an unpredictability concerning the size of the avalanches. Even though the sandpile is driven to the stationary state we still observe that the local slopes (state variables) of the sandpile vary but the local slopes of the pile cannot exceed a specified critical value without a grain of sand tumbling down. A minimally stable site occurs when the local slope equals the critical value. An avalanche is triggered the moment we add a grain of sand that causes the local slope to exceed the critical value which inevitably will happen if our grain of sand provokes a positive change of a minimally stable site. In basic terms, an avalanche sweeps the minimally stable sites that are spatially connected. The variation of the local slopes makes it impossible to predict what will happen when we add a new grain of sand: Either it triggers a global avalanche or the perturbation simply results in small local rearrangements. Occasionally, the additional grain of sand just stays where it fell and no avalanche is produced.
In order to examine the phenomenon of self-organized criticality, BTW introduced a cellular automaton which involves discrete space coordinates [11]. The state of the cellular automaton is fully specified by the values of the physical variables in each site. The dynamical rules for the physical variables determine the evolution of the model. The dynamical rules in the BTW model, at least intuitively, resemble the dynamics of a sandpile: A signal is transmitted from a local site to its nearest neighbors the moment a dynamical integer variable exceeds a critical value.

By simulating this model BTW showed that the system does indeed drive itself to a statistically stationary state, characterized by the distributions of avalanche lifetimes and avalanche sizes which exhibit power-law behavior limited only by the size of the system. Since the system evolves into a stationary state without any characteristic time or length scales, it is in this sense critical and thus it provides a mechanism for establishing a coherence in all time scales. The generic universality of the model stems from the fact that the systems reach the critical attractor without the need to adjust the system parameters. Also, the systems adjust themselves to different environments. In the language of sandpiles: it does not affect the criticality of the final stationary state if we use wet sand instead of dry sand [11].

We stress that the complexity of these models is not a result of complex local rules; they emerge as a result of the continued local interaction between all parts in the extended system. Such examples of dynamical systems which generate complex fractal structures, might provide the explanation of the common appearance of fractal structures in nature. Note that the idea of self-organized criticality (SOC) complements the concept of “chaos” in which simple systems with a few degrees of freedom display quite complex behavior.

The study of the SOC systems has to a great extent been based on simulations that use cellular automata models. The majority of these simulations have been limited to conservative models, i.e., models where the simple dynamical rules conserve the dynamical variable. It has been suggested that the necessary (and sufficient) condition for SOC is indeed the conservation law [15, 16]. The question concerning the presence of nonconservation in these models is essential and related to the generality of such models. P. Bak and C. Tang and others soon recognized a possible relationship between the mechanism of earthquakes and SOC [17, 18, 19, 20]. The conservative SOC models were used as a framework for explaining the power-law behavior of the energies released during earthquakes. But none of the attempts provided the correct power laws and, additionally, they failed to explain the great variability of the power laws. Finally, the conservative SOC models are not related to many basic features of earthquakes, such as the correlations in the occurrence of earthquakes.
Recently we presented a class of nonconservative models that provide those basic features [21, 22, 23]. The new class of models shows a very rich variety of behavior that seem to be related directly to the general 'turbulent' behavior of earthquakes.

The nonconservative model shows self-organized criticality, that is, the probability for energy release during an earthquake is a power law. The model is nonuniversal in the sense that the exponents change continuously as a function of the parameters of the models. A localization transition appears below a certain value of conservation. This model, which can be derived from a very simple model of a seismic system, is an excellent way of looking at the spatiotemporal correlations of the earthquakes, and a great chance to increase our knowledge on the general problems discussed above.

We intend to discuss both the conservative BTW sandpile cellular automaton model and the recently introduced nonconservative model in order to illustrate the different mechanism which drives the systems into a critical dynamical attractor.

In chapter 2 we discuss the BTW sandpile model in the mean-field approximation. In the mean-field model we are able to pinpoint the process of self-organization as well as the emerging scale-invariance displayed as a power-law distribution of avalanche sizes. We show, that the mean-field models of the BTW sandpile model and the forest-fire model are identical.

In chapter 3 we introduce the BTW sandpile model on a lattice and show that the dynamical behavior can be expressed as a transport problem. This implies that the average avalanche size scale with the system size, and that additional heuristic arguments related to the transport properties more than indicate the origin of the power-law behavior. We derive scaling relations between the critical exponents. The scaling relations along with additional constraints leave only one exponent unknown.

Chapter 4 introduces the basic concepts of seismicity, and we show that a very simple spring-block model of an earthquake fault is equivalent to an extended, nonconservative, dynamical system. The system evolves spontaneously into a state characterized by power-law correlations. We interpret the complex dynamical attractor in the following way: An extended system, such as globally coupled oscillators, tends to synchronize, if the interaction is not too weak. This feature is carried through to locally coupled oscillators but the attractor is not robust to noise. Noise will destroy the synchronized system leaving it only partially synchronized. This is the origin of the power-law distributions that have been observed in nonconservative systems.

Chapter 5 reviews some of the interesting features of the spring-block model. We discuss the self-organization process and the nonuniversality of the model displayed through the dependence of the power-law exponent on the level of conservation as well as the boundary conditions. Furthermore, we discuss several measures of the
temporal correlations in the model. The occurrence of earthquakes define a temporal sequence which is multifractal. The correlations vanish when the model becomes conservative. Thus they are intimately related to the nonconservative nature of the model.

In chapter 6 we argue and demonstrate numerically that $1/f$ noise can be explained as a deterministic self-organized critical phenomenon that emerge, without adjustment of the system parameters, in interactive dissipative dynamical systems with many degrees of freedom. We do this by using the very simple dynamical model of earthquake dynamics to capture the fundamental underlying mechanism [24].
The observation that aristocratic families seemed to be more frequently extinct than ordinary families led to the belief that the aristocratic families were less fertile than common people. The study of *branching processes* or *multiplicative processes* proved this hypothesis wrong [25]. Later on, the theory of branching processes appeared in the context of nuclear chain reactions, cosmic-ray showers, spreading of epidemics, and other applications [26].

The branching process is defined as a sequence of random variables \( \{X_n\}_{n=0}^\infty \), in which \( X_n \) represents for instance the number of descendants in the \( n \)'th generation of a family tree with \( P(X_0 = 1) = 1 \). The probability that the first ancestor has \( b \) sons is denoted \( p_b \), \( b = 0, 1, \ldots \) where \( \sum_{b=0}^\infty p_b = 1 \). Each one of his sons has the exact same probability of having \( b \) sons and so on.

![Figure 2.1: A realization of a branching process where the probability of having 0, 1, or 2 sons all equals \( \frac{1}{3} \).](image)
Figure 2.1 shows a member of an ensemble of branching trees where \( p_0 = p_1 = p_2 = \frac{1}{3} \). The future development of different families in a given generation is mutually independent. They do not know the existence of one another; they are uncorrelated.

The study of branching processes provides a highly instructive starting point of the discussion on self-organized criticality in conservative models, because the general features of SOC can be understood in detail in these uncorrelated models. Also, they illustrate some aspects of the disparity between conservative and nonconservative models.

Between uncorrelated models and sandpile models on a regular lattice (for a definition of the latter, see chapter (3) the main difference is numerical rather than conceptual, i.e., the geometry of a finite dimensional lattice (or a lattice below the critical dimension) may lead to different values of the critical exponents, but it will not alter the origin of the observed criticality. The change (if any) in critical exponents arises from the correlations induced by the underlying geometry.

The theory of branching processes is synonymous with the mean-field theory of sandpile models \([27, 28]\); the basic assumption in the latter is the absence of any spatial correlations.

Also, if an upper critical dimension \( d_u \) of the model being investigated exists then the critical indices are the same as the critical indices extracted from the mean-field theory when \( d \geq d_u \). The same exponents can be obtained from the Bethe lattice (Caley tree) which approximates the limit \( d \to \infty \).

The random neighbor model is the most simple way of implementing (approximating) the absence of correlations in branching processes when simulating the model.

### 2.1 The random neighbor model

#### 2.1.1 Definition of the random neighbor model

Given \( N \) sites numbered \( i = 1, \ldots, N \). An integer variable \( z_i \) is associated with every site \( i \). All sites are capable of storing \( z_{th} - 1 \) units. The system is perturbed by adding one unit at a time to a randomly chosen site, that is, \( z_i \to z_i + 1 \). We stop perturbing the system if, at some point, \( z_i \geq z_{th} \): the site topples, i.e., its content is distributed to "neighboring sites" or simply dissipated. To be more precise, if \( z_i \) exceeds the threshold value \( z_{th} \) then the site relaxes: \( z_i \to 0 \), and we add one unit to \( \alpha z_{th} \) randomly chosen neighbors

\[
z_{jk} \to z_{jk} + 1, \quad k = 1, \ldots, \alpha z_{th}. \tag{2.1}
\]
2.1. THE RANDOM NEIGHBOR MODEL

The parameter $\alpha$ determines the number of random neighbors, its form being $\alpha = l/z_{th}$, where $l \in \{0, \ldots, z_{th} - 1\}$.

We can choose new random neighbors every time site $i$ topples, in which case we refer to the model as an annealed random neighbor model. The random choice of neighbors can also remain fixed during the simulations which is then called a random neighbor model with quenched randomness. In the following we will restrict ourselves to the annealed model because it allows the results to be derived very easily.

The dynamics of the model is defined as follows: We take a random initial configuration $\{z_i\}_{i=1}^N, z_i < z_{th} \forall i$. We add one unit at a time to a randomly chosen site $i$, i.e., $z_i \rightarrow z_i + 1$. If $z_i \geq z_{th}$ the system relaxes according to the rule

$$
\begin{align*}
    z_i &\rightarrow 0, \\
    z_{jk} &\rightarrow z_{jk} + 1, \quad k = 1, \ldots, \alpha z_{th}.
\end{align*}
$$

This may cause one or more of the “neighboring sites” to exceed the threshold value, in which case they have to relax simultaneously (i.e., we use a parallel updating of the lattice) according to Eq. (2.2), and we say that an avalanche propagates in the system. This process continues until we regain a static state $\{z_i\}_{i=1}^N, z_i < z_{th} \forall i$. Then we perturb the system once again and so on.

2.1.2 The statistically stationary state

Let $P_z$ be the probability that a given site contains $z$ units. One toppling can cause 0 to $\alpha z_{th}$ new topplings. It will cause a neighbor to topple if the neighbor has $z_{th} - 1$ units. Since the neighbors are randomly chosen, it will cause $b$ topplings with the probability

$$
p_b = \binom{\alpha z_{th}}{b} P_{z_{th} - 1}^b (1 - P_{z_{th} - 1})^{\alpha z_{th} - b}, \quad b = 0, \ldots, \alpha z_{th}
$$

averaging the amount of new topplings to

$$
\langle b \rangle = \sum_{b=0}^{\alpha z_{th}} b p_b = \alpha z_{th} P_{z_{th} - 1}.
$$

If $\alpha < 1$ and $N \rightarrow \infty$ this model does not form loops (the probability that a toppling site is chosen as a neighbor by one of its own neighbors is of the order of $1/N$), and the random neighbor model is considered a true branching process. The probability of creating $b$ new branches is given by Eq. (2.3), and on the average $\langle b \rangle = \alpha z_{th} P_{z_{th} - 1}$.
branches are created. Very quickly the system settles into a statistically stationary
state in which \( \langle z \rangle = \frac{1}{N} \sum_{i=1}^{N} z_i \) fluctuates around a constant value.

Note that when \( \alpha = 1 \) the random neighbor model is not well-defined since \( \langle z \rangle \)
will continue to grow until \( \langle z \rangle = z_{th} \). The following perturbation will initiate an
avalanche which goes on forever. However, the branching process is mathematically
well-defined.

As an illustration of the results we have simulated an annealed random neighbor
system where \( N = 50000 \) and \( z_{th} = 20 \). Figure 2.2 shows how, very quickly, the
system reaches the statistically stationary state in which the measured mean value of
the dynamical variable \( z \) per site fluctuates (the system is finite) around the average
value \( \langle z \rangle \).

\[
\langle z \rangle = \frac{1}{N} \sum_{i=1}^{N} z_i
\]

Figure 2.2: The average value \( \langle z \rangle = \frac{1}{N} \sum_{i=1}^{N} z_i \) as a function of the observed number
of avalanches. (a) The number of chosen neighbors in the random neighbor model
equals two, i.e., \( \alpha z_{th} = 2 \). (b) The number of chosen neighbors is 18.

It is easily shown that the system reaches an equilibrium state, where the rate
of flow into a state \( z \) equals the rate of flow out of that same state. During an
avalanche (or perturbation) the rate of flow into the state \( z \) is proportional to the
probability that a chosen site contains exactly \( z - 1 \) units, i.e., \( P_{z-1} \), while the rate
of flow out of the state \( z \) is proportional to \( P_z \) (the constants of proportionality are
the same, namely the total number of toppling lattice sites at that moment). Thus

\[
P_{z-1} = P_z, \quad z = 1, \ldots, z_{th} - 1
\]

\[
(2.5)
\]
is an attractor of the system. Using the normalization condition \( \sum_{z=0}^{z_{th}-1} P_z = 1 \), we
find that

\[
P_z = \frac{1}{z_{th}}, \quad z = 0, \ldots, z_{th} - 1.
\]

\[
(2.6)
\]
We notice that the organization of the system is independent of \( \alpha \) because the derivation of Eq. (2.6) does not rely on the value of \( \alpha \). This separation of the dynamics of the avalanches from their organization of the medium through which they propagate, is probably the reason why it is possible to obtain analytical results.

Figure 2.3 displays a temporal average of \( P_z \), the average being measured just before an avalanche is triggered. Note that \( \{P_z\}_{z=0}^{z_{th}-1} \) is constant despite different values of \( \alpha \).

![Figure 2.3: The distribution function of states \( P_z \) when (a) \( \alpha = \frac{2}{20} \) and (b) \( \alpha = \frac{18}{20} \). The medium is independent of \( \alpha \) in accordance with Eq. (2.6).](image)

Using Eq. (2.4) we find that the average amount of new topplings is determined by the parameter \( \alpha \)

\[
\langle b \rangle = \alpha.
\]  

Furthermore, the average \( z \)-value per site in the system is given by

\[
\langle z \rangle = \sum_{z=0}^{z_{th}-1} z P_z = \frac{z_{th} - 1}{2},
\]

see Figure 2.2.

### 2.1.3 The avalanche size distribution

The parameter \( \alpha \) determines the amount of nonconservation in the system. If \( \alpha < 1 \) we dissipate \( z_{th} - \alpha z_{th} \) units in every toppling. In a statistically stationary state we must dissipate as many units as we put into the system. Let \( s \) denote the size of an avalanche (i.e., the total number of topplings in an avalanche) and let \( \langle s \rangle \) be the
average size of an avalanche. Then

\[ \langle s \rangle \cdot (z_{th} - \alpha z_{th}) = \frac{1}{P_{z_{th}-1}} = z_{th} \]  \hspace{1cm} (2.9)

since 1/P_{z_{th}-1} is the average rate of flow into the system (the number of additions before we trigger an avalanche). Thus the average avalanche size is given by

\[ \langle s \rangle = \frac{1}{1 - \alpha}, \]  \hspace{1cm} (2.10)

see Figure 2.4.

Figure 2.4: The measured average size of an avalanche \( \langle s \rangle \) as a function of \( 1/(1 - \alpha) \) illustrates the linear dependence in Eq. (2.10).

In fact, it is possible to express the distribution function of avalanche sizes \( s \) analytically, see e.g. Ref. [26]

\[ P(S = s) \sim s^{1-\tau_s} \cdot \exp\left(\frac{s}{\xi}\right) = s^{-\frac{3}{2}} \cdot \exp\left(-\frac{s}{\xi(\alpha)}\right), \]

\[ \xi(\alpha) \sim \frac{1}{(1 - \alpha)^2}, \]  \hspace{1cm} (2.11)

where we introduce the power-law exponent \( \tau_s \) and the correlation length \( \xi \). Note that Eq. (2.11b) along with with the left-hand side of Eq. (2.11a) implies that \( \tau_s = \frac{5}{2} \).

When \( \alpha = 1 \) the model is conservative and \( \langle b \rangle = 1, \langle s \rangle = \infty \), and \( \xi = \infty \), following that \( P(S = s) \sim s^{-\frac{3}{2}} \); the system is critical. When \( \alpha < 1 \) the model is nonconservative and \( \langle b \rangle = \alpha < 1, \langle s \rangle = 1/(1 - \alpha) < \infty \), and \( \xi < \infty \), following that \( P(S = s) \sim s^{-\frac{3}{2}} \cdot \exp(-\frac{s}{\xi}) \); the system is subcritical, see e.g. [29].

Figure 2.5(a) are graphs of the distribution function of avalanche sizes, obtained from our test simulations. We have fitted the measured distribution function to the
form given by Eq. (2.11) and plotted the correlation length $\xi$ against $1/(1 - \alpha)^2$ in Figure 2.5(b).

![Graphs showing the correlation length $\xi$ as a function of $1/(1 - \alpha)^2$](image)

Figure 2.5: (a) The measured distribution function of avalanche sizes in the annealed random neighbor model with 2 and 18 neighbors, respectively. The solid lines show the analytical expression of Eq. (2.11a) when we use $\xi = 0.64$ and $\xi = 180$, respectively. (b) The correlation length $\xi$ as a function of $1/(1 - \alpha)^2$.

### 2.1.4 The interpretation of the parameter $\alpha$

Finally, let us approach the random neighbor model from a slightly different angle. Let $P$ be the probability that a given neighbor is a boundary site where, by definition, added sand simply dissipates. Let the relaxation dynamics be defined by

$$
\begin{align*}
  z_i & \to 0, \\
  z_{jk} & \to z_{jk} + 1, \quad k = 1, \ldots, z_{th}.
\end{align*}
$$

If the randomly chosen neighbor $j_k$ is a boundary site with probability the $P = 1 - \alpha$, then $\alpha \in [0, 1]$ is a continuous variable. We can immediately apply all the previous results. Only the interpretation of $\alpha$ differs slightly: $\alpha$ conveys the probability that a randomly chosen neighbor is not a boundary site. On average every toppling dissipates

$$
  z_{th} - \alpha z_{th} = (1 - \alpha)z_{th}
$$

units. Thus, we define $\alpha$ as the level of conservation in the model. In the limit $\alpha \to 1$ the system will be critical with $P(S = s) \sim s^{1-\tau_s}$ where $\tau_s = \frac{5}{2}$. 

CHAPTER 2. THE BRANCHING PROCESS

2.2 Models within the same universality class

The exponent $\tau_s = \frac{5}{2}$ appears in several other models. Thus, these models belong to the same universality class as branching processes. One example being the sandpile model on the Bethe lattice [30, 31]. This fact should not surprise the reader as the Bethe lattice approximates the limit of very large dimensions. We notice that analytical results can drawn from the Bethe lattice too because, once again, the dynamics of the model separates from the medium. Recollections of correlations exist between the lattice sites in the Bethe lattice, but they decrease exponentially with the distance $r$ between the sites and do not affect the value of the critical indices [30]. The correlation does however affect the attractor of the Bethe lattice which is quite different from that of the branching process. In the case of a Bethe lattice with coordination number 3 (and $z_{th} = 2$) D. Dhar and S. N. Majumdar have proven that $P_0 = 1/12$, $P_1 = 4/12$, and $P_2 = 7/12$ [30]. However, the correlations in a regular $d$-dimensional lattice only decrease like $r^{-2d}$, and hence change the critical indices [32]. Figure 2.6 shows, how the power-law exponent in the BTW model changes with dimension $d$. The exponent $\tau_s$ reaches the mean-field value of $5/2$ in $d = 5$, thus indicating that the critical dimension $d_u = 5$.

![Figure 2.6](image-url)

Figure 2.6: (a) The distribution of avalanche sizes in the BTW model in $d = 2, 3, 4$ and 5. The slope increases with dimension. (b) The power-law exponent approaches the mean-field value when the dimension increases. When $d = 5$ we cannot distinguish the measured power-law exponent from $5/2$.

When $d \geq d_u = 3$ the directed model of D. Dhar and R. Ramaswamy gives the value $\tau_s = \frac{5}{2}$ [33]. In $d = 2$ they find that $\tau_s = \frac{7}{4}$.

Another model in the same category, of which the branching process is a white-washed representative, is the model presented by S. S. Manna et al. [34]. They
debate a model where the local dynamics is nonconservative though the model is
globally conservative. They define a dynamical rule with global conservation, and
destroying the spatial correlations by hand they let the toppling site $z_i \to 0 + \theta$,
where $\theta \in \{-3, \ldots, 3\}$ is an annealed random variable. As a result the site $i$ becomes
a random neighbor to its own neighbors. They obtain $\tau_s = 2.515 \pm 0.020$ when their
data is extrapolated to $N \to \infty$.

2.3 Mean-field theory of the forest-fire model

Recently B. Drossel and F. Schwabl [35] introduced a new critical version of the
forest-fire model that was originally introduced by P. Bak et al. [36]. The model is
defined in a $d$-dimensional hypercubic lattice of linear size $L$. A lattice site is either
empty, a tree, or a burning tree. At every time-step the forest-fire model evolves
according to the following rules: A tree grows in an empty site with probability $p).
A site with a burning tree becomes an empty site. A tree becomes a burning tree
if no less than one of its nearest neighbors is a burning tree whilst a tree without
burning neighbors catches fire spontaneously with the probability $f$.

We discuss the model in the mean-field language, that is, we disregard any
correlations between the lattice sites. Let $\rho_e(\tau)$, $\rho_t(\tau)$, and $\rho_f(\tau)$ denote the densities
of empty sites, trees, and fires at time $\tau$. The mean-field equations for the forest-fire
model are given by the rate equations along with the normalization constraint

\begin{align*}
\rho_e(\tau + 1) &= \rho_e(\tau) + \rho_f(\tau) - pp_e(\tau), \\
\rho_t(\tau + 1) &= \rho_t(\tau) - f\rho_t(\tau) + pp_e(\tau) - m\rho_f(\tau)\rho_t(\tau), \\
\rho_f(\tau + 1) &= f\rho_t(\tau) + m\rho_f(\tau)\rho_t(\tau), \\
\rho_e(\tau) + \rho_t(\tau) + \rho_f(\tau) &= 1, \quad (2.14)
\end{align*}

in which $m$ denotes the number of neighbors.

The system gradually develops into a statistically stationary state. With many
trees in the forest the fire will more easily propagate in the system resulting in
more burning trees than growing trees. With few trees in the forest the fire will
more easily get extinguished with more growing trees than burning trees as a result.
The densities are independent of time in the the stationary state, i.e., $\rho_i(\tau + 1) =
\rho_i(\tau) = \rho_i$ for $i \in \{e, t, f\}$. The rate equations and the normalization constraint can
be simplified to

\begin{align*}
0 &= \rho_f - pp_e, \\
0 &= -f\rho_t + pp_e - m\rho_f\rho_t,
\end{align*}
CHAPTER 2. THE BRANCHING PROCESS

\[
\begin{align*}
\rho_f &= f \rho_t + m \rho_f \rho_t, \\
\rho_t + \rho_e + \rho_f &= 1. 
\end{align*}
\]  

(2.15)

This set of equations is equivalent to

\[
\begin{align*}
\rho_f &= p \rho_e, \\
\rho_e &= \frac{1 - \rho_t}{1 + p}, \\
m \rho_t^2 &= (m + 1 + \frac{f(1+p)}{p}) \rho_t + 1 = 0.
\end{align*}
\]  

(2.16)

Thus we have 3 independent equations to determine the densities. Let

\[
k = \frac{f(1+p)}{p}.
\]  

(2.17)

Solving Eq. (2.16c) we find

\[
\rho_t = \frac{m + 1 \pm \sqrt{(m-1)^2 + 2(m+1)(k+k^2)}}{2m}
\]

\[
= \frac{1}{m} - \frac{1}{m^2 - m} k + O(k^2),
\]  

(2.18)

where we expanded the density of trees in powers of \( k \). Note that only the solution containing the minus sign is physical, i.e., \( \rho_t < 1 \).

One burning tree can cause \( 0 \) to \( m \) new trees to burn. If a chosen neighbor is a tree the neighbor will be set on fire. The average amount of new burning trees is

\[
\langle b \rangle = \sum_{b=0}^{m} b \binom{m}{b} \rho_t^b (1 - \rho_t)^{m-b}
\]

\[
= m \rho_t
\]

\[
= m \left( \frac{1}{m} - \frac{1}{m^2 - m} k + O(k^2) \right)
\]

\[
= 1 - \frac{1}{m-1} k + mO(k^2).
\]  

(2.19)

If \( k \to 0 \) then \( \langle b \rangle \to 1 \), and the mean-field theory of the forest-fire model will be nothing but a critical branching process. If \( k \neq 0 \) then \( \langle b \rangle < 1 \) and the system is subcritical. The parameter \( 1 - k/(m-1) \) is analogous to the parameter \( \alpha \) in the random neighbor model.

B. Drossel and S. Schwabl claims that the power-law exponent \( \tau_s = 2 \), independent of dimension \( d \). If a mean-field description of the forest-fire model exists then the distribution function of the number of burned trees initiated by a single fire
2.4. SUMMARY

exhibits a power-law behavior with an exponent \( \tau_s = 5/2 \). Intuitively, the forest-fire model should converge to the random neighbor approximation in the limit \( d \to \infty \). Preliminary simulation results of the forest-fire model shows, that the power-law exponent does indeed change with dimension, but the rate of change is small. We find \( \tau_s = 2.11, 2.24, 2.31, \) and \( 2.38 \) in 2, 3, 4, and 5 dimensions, respectively.

2.4 Summary

Models without any spatial correlations are described as branching processes equivalent to the mean-field description. We introduced the random neighbor models in order to be able to simulate the mean-field theory. These models evolve naturally – self-organize – into a statistically stationary state where the distribution of avalanche sizes is given by

\[
P(S = s) \sim s^{1-\tau_s} \cdot \exp\left(-\frac{s}{\xi(\alpha)}\right), \\
\xi(\alpha) \to \infty \text{ for } \alpha \to 1.
\] (2.20)

The parameter \( \alpha \) can be interpreted as being related to the dissipation in the model. If \( \alpha = 1 \) the model is conservative, and the system is critical since the distribution function in Eq. (2.20) shows a pure power law. Any degree of nonconservation \( (\alpha < 1) \) will introduce a finite correlation length. Different models prove to be in the same universality class as the random neighbor model.

Finally, we have argued, that the mean-field description of the forest-fire model introduced by B. Drossel and F. Schwabl is just a branching process, where \( k/(m-1) \) is equivalent to \( 1 - \alpha \). When \( k \to 0 \), the branching process becomes critical. This implies that the distribution function of the number of burned trees in a single forest fire in the random neighbor approximation is a power-law distribution with exponent \( \tau_s = 5/2 \). Preliminary simulations support this conjecture which contradicts the claim that the power-law exponent \( \tau_s = 2 \) in all dimensions.
Chapter 3

Models with a spatial structure

3.1 Definition of the model

Let the geometry of the model (i.e., the neighborhood relations) be given by an underlying lattice. We assign an integer $z_i$ to each lattice site $i$, where $i = 1, \ldots, N$. The integer $z_i$ represents an appropriate dynamical variable (e.g., mechanical stress, heat, pressure, or energy) in site $i$ in a spatially extended system. In the following we refer to $z_i$ as the height of a column of sand in site $i$. We perturb the system (add sand to the system) by choosing at random a position increasing the dynamical variable with one unit, i.e., $z_i \rightarrow z_i + 1$.

Whenever the dynamical variable in site $i$ exceeds a threshold value, the site topples. Let $C$ denote the coordination number of an interior point in the lattice, and let $i_k, k = 1, \ldots, C_i$ denote the nearest neighbors of site $i$ with the coordination number $C_i$. For reasons of simplicity, we choose $z_{th} = C$ in the following. It implies that the allowed values in site $i$ are $0, \ldots, z_{th} - 1$. The process of a toppling in site $i$ is defined by

\[
\begin{align*}
  z_i & \rightarrow z_i - C, \\
  z_{i_k} & \rightarrow z_{i_k} + 1, \quad k = 1, \ldots, C_i.
\end{align*}
\]  

As a result one or more neighbors may exceed the threshold value in which case they have to relax and an avalanche will propagate in the system. This model is defined as undirected since the local toppling rule is an isotropic mechanism. We describe the model as being directed if the toppling is anisotropic [37].

The toppling rule conserves the amount of $z$-values whenever an interior site topples ($C_i = C$). Dissipation only occurs when a boundary site topples ($C_i < C$) because a boundary site has fewer nearest neighbors than a site within the system.
CHAPTER 3. MODELS WITH A SPATIAL STRUCTURE

3.1.1 The attractor of the system

Analogues to the behavior of the random neighbor model this system will reach an equilibrium state where, on the average, the rate of flow into the system equals the rate of flow out of the system across the boundary. That is, \( \langle z \rangle = \frac{1}{N} \sum_{i=1}^{N} z_i \) fluctuates around a constant value. The system is very sensitive to perturbations if \( \langle z \rangle \) becomes too big. Sand will rapidly be transferred to the boundary, where it dissipates. On the other hand, the system is not as sensitive to perturbations if \( \langle z \rangle \) becomes too small. These two situations – to maximize the average \( z \)-value and to stabilize the system by dissipating sand at the boundary – maintain an equilibrium situation. L. Pietronero et al. have shown that it is possible to calculate \( \langle z \rangle \) by using an “effective medium theory” which relies on the above mentioned balance between the rate of flow into and out of the system [38].

Unfortunately we cannot apply the same argument as to the random neighbor model leading to a statistically stationary state characterized by \( P_0 = \ldots = P_{z_{th} - 1} \), since the proof relies on an assumption of totally uncorrelated sites.

3.2 The implications of the spatial structure

The introduction of the spatial structure of the model has two implications. First, the spatial correlations will introduce correlations between the values in the lattice sites. That is, not all of the \( z_{th} \times N \) stable configurations are allowed when the system has reached the equilibrium state. E.g., two nearest neighbors cannot both be equal to 0. Suppose \( z_i = 0 \). The site has just relaxed following that all the nearest neighbors \( z_{ik} \geq 1, k = 1, \ldots, C_i \) since one unit is transferred to each of the nearest neighboring sites by the relaxation rule.

It is possible to get more quantitative results. S. N. Majumdar and D. Dhar have analytically calculated the probability that two sites separated by a distance \( r \) in a \( d \)-dimensional hypercubic lattice would both have the minimum value (0 in our definition) [32]. They find an anticorrelation which, to the lowest order in \( r \), varies as \( r^{-2d} \) for large \( r \). In the case of the Bethe lattice the anticorrelations decrease exponentially as D. Dhar and S. N. Majumdar show [30]. In truly infinite dimensions (the branching process) these correlations vanish completely, by definition.

Second, the question of dissipation is transformed into a transport problem where the input has to be transported from neighbor to neighbor until it reaches a site where it dissipates – in this case at the boundary. If no dissipation takes place at all (e.g., if the model is defined with periodic boundary conditions) the model is not well-defined as was the case with the random neighbor model.
3.3 Scaling arguments

3.3.1 Scaling of the average avalanche size

In the following we restrict the discussion to \( d \)-dimensional hypercubic lattices of linear size \( L \) \((N = L^d)\). The system can only dissipate energy through the boundary. The average distance a particle has to travel to reach the boundary is proportional to \( L \) when we deposit the units at random. This implies that

\[
\langle s \rangle \geq K(L) \cdot L,
\]

where \( K(L) \) is an \( L \)-dependent function \((\frac{dK}{dL} \geq 0)\). The specific form of the function \( K \) depends on the actual relaxation dynamics. Thus the average size of avalanches scales with system size and is infinite in the thermodynamic limit. If the particle diffuses out to the boundary then \( K(L) = C \cdot L \), where \( C \) is a constant. Thus

\[
\langle s \rangle \geq C \cdot L^2.
\]

The added particles have to perform \( L^2 \) steps to cover the distance \( L \) as pointed out by L. P. Kadanoff et al. [39]. This is indeed the case for the BTW sandpile relaxation given by Eq. (3.1). D. Dhar has given an analytical proof that \( \langle s \rangle \sim L^2 \) in the undirected BTW-type relaxations independent of dimension [40]. This has been verified by measurements of P. Grassberger and S. S. Manna [31].

In a directed model \( \langle s \rangle \sim L \) so \( K(L) = \tilde{C} \), see for example Ref. [39]. This can be shown analytically by copying the proof of D. Dhar given for the undirected version of the model.

Let \( P(S = s) \) denote the distribution function of avalanche sizes. By definition

\[
\langle s \rangle = \int s P(S = s) \, ds
\]

following that \( P(S = s) \) must be of a form producing an infinite average value in the thermodynamic limit \( L \to \infty \). Motivated from the discussion of branching processes (and from percolation theory among others) we postulate a distribution function of the form

\[
P(S = s) \sim s^{1 - \tau_s} \cdot \exp(-\frac{s}{\xi(L)}),
\]

The distribution function cannot decrease exponentially in the thermodynamic limit since this will make the average avalanche size finite. A pure power-law behavior, however, produces a divergence of the average avalanche size (provided the power-law exponent is smaller than 3). Thus the correlation length has to go to infinity when \( L \to \infty \) to assure that the average value becomes infinite:

\[
\xi(L) \to \infty \quad \text{when} \quad L \to \infty.
\]
Only in exceptional cases is it possible to prove that the distribution function is a power law in the thermodynamic limit: the Bethe lattice where $\tau_s = \frac{5}{2}$ and the directed model with $\tau_s = \frac{7}{3}$ in $d = 2$ and $\tau_s = \frac{5}{2}$ in $d \geq 3$. In all other models we have to rely on simulations in order to convince possible sceptics of the power-law behavior of the distribution function of avalanche sizes.

Because we are unable to prove the existence of power-law behavior we are also unable to get analytical expressions for the power-law exponent. However, if we assume a power-law behavior we can obtain some limits for the power-law exponent $\tau_s$ as well as relate the critical exponent to other critical exponents yet to be defined.

3.3.2 Finite-size scaling and scaling relations

Let $P(S = s, L)$ denote the distribution function in a system with linear size $L$. Suppose that the distribution function is a power law up to a certain cutoff size which depends on the system size $L$. If this dependence is of a power-law type, then a generalization of Eq. (3.5) is the finite-size scaling ansatz

$$P(S = s, L) \sim L^{-\beta} \cdot g\left(\frac{s}{L^\nu}\right),$$

where $g$ is a universal scaling function and $\beta$ and $\nu$ are critical indices. $\nu$ describes how the cutoff size scales with system size. By rewriting the finite-size scaling ansatz

$$P(S = s, L) \sim L^{-\beta} \cdot \left(\frac{s}{L^\nu}\right)^{-\frac{\beta}{\nu}} \cdot \tilde{g}\left(\frac{s}{L^\nu}\right)$$

we easily see that it is a generalization of the former ansatz in Eq. (3.5). As a by-product, we get a scaling relation, i.e., a relation between the three critical exponents

$$1 - \tau_s = -\frac{\beta}{\nu}. \quad (3.9)$$

We can take advantage of the knowledge that for undirected models $\langle s \rangle \sim L^2$, while for directed models $\langle s \rangle \sim L$ – valid in all dimensions – to get an additional scaling relation

$$\langle s \rangle = \int_{1/L^\nu}^{\infty} s P(S = s, L) ds / \int_1^{\infty} P(S = s, L) ds$$

$$= \int_{1/L^\nu}^{\infty} s L^{-\beta} \cdot g\left(\frac{s}{L^\nu}\right) ds$$

$$= L^{2\nu - \beta} \int_{1/L^\nu}^{\infty} \tilde{g}(\tilde{s}) d\tilde{s}$$

$$\rightarrow L^{2\nu - \beta} \quad \text{for} \quad L \rightarrow \infty. \quad (3.10)$$
3.3. **SCALING ARGUMENTS**

To perform the last step we assume that the integral converges. Therefore we have

\[
\begin{align*}
2\nu - \beta &= 2 \quad \text{undirected}, \\
2\nu - \beta &= 1 \quad \text{directed}.
\end{align*}
\] (3.11)

Eliminating \(\beta\) using Eq. (3.9) we get

\[
\begin{align*}
2 &= \nu(3 - \tau_s) \quad \text{undirected}, \\
1 &= \nu(3 - \tau_s) \quad \text{directed},
\end{align*}
\] (3.12)

which once again shows that \(\tau_s \leq 3\) since \(\nu \geq 0\).

We can induce further scaling relations between additional critical exponents. Let \(S, T, R, A\) denote the stochastic variables corresponding to the size, lifetime, radius (linear size), and area of the avalanches, respectively. We assume a power-law behavior of the probability densities, that is

\[
P(X = x) \sim x^{1 - \tau_x},
\] (3.13)

where \(X \in \{S, T, R, A\}\) and \(x \in \{s, t, r, a\}\). The existence of a relation between the different stochastic variables implies the existence of scaling relations between the exponents \(\tau_s, \tau_t, \tau_r, \) and \(\tau_a\). Let \(\gamma_{xy}\) denote the exponent relating \(X\) with \(Y\), i.e.,

\[
X = Y^{\gamma_{xy}}.
\] (3.14)

Thus

\[
P(X = x) \, dx = P(Y = y) \, dy \quad \Rightarrow \quad x^{1 - \tau_x} \sim y^{1 - \tau_y} \frac{dy}{dx} \Rightarrow \quad x^{1 - \tau_x} \sim \frac{1}{x^{\gamma_{xy}}} \cdot x^{\frac{1}{\gamma_{xy}} - 1}
\] (3.15)

resulting in

\[
\tau_x = 2 + \frac{\tau_y - 2}{\gamma_{xy}}.
\] (3.16)

According to Eq. (3.14) \(Y = X^{\gamma_{xy}}\) so by definition

\[
\gamma_{xy}^{-1} = \gamma_{yx}.
\] (3.17)

Also, we find that

\[
X = Y^{\gamma_{xy}} = (Z^{\gamma_{yz}})^{\gamma_{xy}} \iff \gamma_{xy} \gamma_{yz} = \gamma_{xz}.
\] (3.18)
The exponents $\gamma_{xy}$ can be measured as well. They appear as the exponents of the conditional expectation values \[41\]. For example

$$E[S \mid T = t] \overset{\text{def}}{=} \sum_s sP(S = s \mid T = t) \sim t^{\gamma_{st}} \tag{3.19}$$

where $P(S = s \mid T = t)$ is the conditional density of $S$ given $T = t$. The reciprocal relationship between the exponents relating the different stochastic variables is a necessary condition if we assume the existence of a transformation between the stochastic variables. In a strictly mathematical sense such transformations cannot exist since, as an example, there will always be more than one possible avalanche size $s$ for a given lifetime $t$. However, if we find that the reciprocal relationships are fulfilled quite accurately, it indicates that the conditional densities, e.g. $P(S = s \mid T = t)$, have a narrow support around their average value.

With $X \in \{S, T, R, A\}$ we have 16 unknown altogether, namely $\tau_x$ and $\gamma_{xy}$ where $x, y \in \{s, t, r, a\}$ with $x \neq y$, but there exist only 12 linearly independent equations in the form of Eq. (3.16) and Eq. (3.17). Eq. (3.18) does not contain any new information, however, we can add two additional equations.

### 3.3.3 Scaling exponent determined by compactness and isotropy

It is quite easy to prove that the avalanches are compact. No holes are allowed inside an avalanche if the system has reached the critical state. We apply the concept of forbidden subconfiguration introduced by D. Dhar to prove the compactness \[40\]. Let $z_{th} = C$, where $C$ is the coordination number of an interior lattice site. Any set $\mathcal{F}$ with $M$ heights satisfying

$$z_i \leq C_i - 1 \quad \forall z_i \in \mathcal{F} \tag{3.20}$$

is called a forbidden subconfiguration. A configuration $\{z_i\}$ is an allowed subconfiguration only if it does not contain any forbidden subconfigurations. A system is in the critical state if and only if it is an allowed configuration \[40\].

The characterization of an allowed state enables us to define an algorithm – known as the burning algorithm – to determine whether a given state is critical. Let $T$ be a “test” state. $T$ is a forbidden state if all sites $i$ in $T$ satisfy $z_i \leq C_i - 1$. Otherwise, there are some sites $j$ in which $z_j > C_j - 1$. Let $T^{(1)}$ denote the smaller set we obtain when deleting (burning) all these sites. If possible we repeat the burning process in $T^{(1)}$, but at some point we cannot continue the burning process: The final set $T^{(n)}$ is either an empty set, in which case $T$ is a critical state, or a nonempty set, in which case $T$ is not a critical state.
First, we prove that avalanches are compact in a 2-dimensional square lattice with $z_{th} = 4$. Suppose that we do have an avalanche with holes inside. Let $S$ be a connected part without any topplings inside the avalanche and let $\tilde{S}$ denote the same set just before the avalanche. If $N_{en}$ denotes the number of exterior neighbors then the boundary sites of $\tilde{S}$ satisfy the inequality

$$\tilde{z}_b \leq 3 - N_{en}$$

(3.21)
since they do not topple, see Figure 3.1.

![Figure 3.1](image)

Figure 3.1: To prevent topplings inside $S$ the boundary sites of $\tilde{S}$ cannot be larger than 3 minus the number of exterior neighbors. The figure displays a situation in which the boundary sites equal the maximum value. By inspection we see that the configuration $\tilde{S}$ is a forbidden subconfiguration, i.e., $\tilde{S}$ is not burnable.

Let $C_i$ denote the coordination number of sites within $\tilde{S}$. Using

$$C_i + N_{en} = 4$$

(3.22)
the boundary sites of $\tilde{S}$ satisfy

$$\tilde{z}_b \leq C_i - 1.$$  

(3.23)

The sites $\tilde{z}$ in the interior of $\tilde{S}$ also fulfill the condition $\tilde{z} \leq C_i - 1$ so $\tilde{S}$ is a forbidden subconfiguration. Thus the system was not in the critical state when the avalanche was initiated.

We can prove the compactness of avalanches in an arbitrary lattice by use of the principle of induction: If $S$ consists of just one site $\tilde{S}$ is not an allowed subconfiguration since

$$\tilde{z} \leq z_{th} - 1 - N_{en} = -1.$$  

(3.24)
Assume that if $S$ consists of $M$ sites then $\tilde{S}$ is not an allowed subconfiguration. Then $S$ cannot consist of $M+1$ sites either, since $\tilde{S}$ will not be an allowed subconfiguration. Adding one site to a set which is not allowed before hand does not make it an allowed set.

Using the compactness along with the isotropic nature of the avalanches we conclude that
\begin{equation}
\gamma_{ar} = 2, \tag{3.25}
\end{equation}
but we are still short of three equations in order to determine all the critical exponents.

### 3.3.4 Scaling relation induced by central site multiple topplings

S. N. Majumdar and D. Dhar supply us with an additional equation [42]. Using the analytical result that the average number of topplings at the site initiating the avalanche $\langle n_c \rangle$ scales with the linear system size $L$ as
\begin{equation}
\langle n_c \rangle \sim \log_{10}(L), \tag{3.26}
\end{equation}
and assuming that
\begin{equation}
S \sim A \cdot n_c \tag{3.27}
\end{equation}
it is possible to show that
\begin{equation}
\gamma_{sr} = \tau_r. \tag{3.28}
\end{equation}
Substituting Eq. (3.28) into Eq. (3.16) with $(x,y) = (s,r)$ we find that
\begin{equation}
\tau_s = 2 + \frac{\tau_r - 2}{\tau_r} \Leftrightarrow 2 = \tau_r(3 - \tau_s), \tag{3.29}
\end{equation}
showing that Eq. (3.28) is equivalent with $\nu = \tau_r$, see Eq. (3.12). Thus we are “only” short of two equations.

### 3.3.5 The dynamical exponent $\gamma_{tr}$

A possible way out of the shortfall of equations is either to establish a connection between the BTW model and other models in which some of the exponents are known analytically or to use additional information about the model (for example the transport properties) to derive further constraints on the critical indices.

S. N. Majumdar and D. Dhar have shown that each allowed configuration in the sandpile model corresponds to a spanning tree [42]. Taking advantage of the known relationship between the latter and the $q$-state Potts model they establish
3.3. SCALING ARGUMENTS

an equivalence between the sandpile model and the \( q \to 0 \) limit of the \( q \)-state Potts model.

The dynamical exponent \( \gamma_{tr} \) in a spanning tree equals \( 5/4 \). Majumdar and Dhar use this value as the value of the dynamical exponent in the BTW model. Thus they only have to measure one exponent to determine all the others. They use the ansatz \( \tau_a = 13/6 \) in order to calculate the remaining critical exponents. We list the findings in Table 3.1, see page 33.

However, the dynamical exponent in the spanning tree reflects, how a spanning tree burn using the burning algorithm. Thus the dynamical exponent \( \gamma_{tr} = 5/4 \) is related to a (static) geometrical description of a spanning tree but has nothing to do with the dynamics of getting from one critical state of the sandpile to another.

Y.-C. Zhang has proposed \( \gamma_{tr} = 4/3 \) based on heuristic arguments [43]. The idea is to view the avalanche as a collection of activation fronts that propagates in the lattice. The activation front at time \( \tau \) is defined as the toppling sites at time \( \tau \). If the activation front was to perform a pure random walk

\[
T \sim R^{\gamma_{tr}} = R^2.
\]

However, sites which have just toppled tend to repel the activation fronts. Thus the avalanche prefers to expand outward and we we expect \( \gamma_{tr} < 2 \). If the activation front perform a self-avoiding random walk it implies

\[
\gamma_{tr} = \frac{4}{3}.
\]

Y.-C. Zhang also suggests that \( \tau_r = 2 \). This is not consistent with the value of \( \tau_s = \frac{15}{7} \) given in Ref. [38] if we take the scaling relations into account.

3.3.6 The interpretation of the fractal boundary

The following argument shows that the critical exponent \( \tau_r \) is related to the fractal properties of the boundary of an avalanche. We do not determine the exponent \( \tau_r \) but the argument explains why it is larger than 2.

Assume we perturb a critical system in the center of a fixed circle of radius \( r \). On the average, the rate of flow through the surface of the circle must be a constant. For simplicity we make the additional assumption that the avalanche is circular. It is clear that only avalanches with a radius larger than or equal to \( r \) can induce a transport through the surface of the circle.

Let us assume we have a situation where the sites only topple once during an avalanche, i.e., the area of the avalanche is identical to the size of the avalanche. The only transport (change of \( z \)-values) in such an avalanche is at the avalanche
surface. Schematically, the situation is shown in Figure 3.2 where it is disclosed that the amount of transport is proportional to the length of the boundary.

Figure 3.2: An avalanche in which the sites topple exactly once only the sites at the boundary change $z$-value. The graph represents the integrated amount of change in $z$-value $\int_{C(\tilde{r})} \Delta z \, dl$ in an avalanche of linear size $r$, where $C(\tilde{r})$ is the circumference of radius $\tilde{r}$.

Thus

$$P(R = r) \cdot r \sim 1 \Rightarrow P(R = r) \sim r^{-\tau_r}$$

(3.32)

in accordance with the suggestion of Y.-C. Zhang [43]. This argument may, by the way, indicate why power-law behavior is observed in the sandpile models.

If the avalanches are compact Eq. (3.25) but the boundary is a fractal, i.e., the length of the boundary is $r^D$, then

$$P(R = r) \cdot r^D \sim 1 \Rightarrow P(R = r) \sim r^{-D}.$$  

(3.33)

Thus

$$1 + D = \tau_r,$$

(3.34)

that is, the exponent $\tau_r$ is intimately linked with the fractality of the boundary.

P. Grassberger and S. S. Manna have measured the fractal dimension of the boundary to $D = 1.22$ [31]. This value agrees with our measurements of the exponent $\tau_r$.

Note that using the result from Eq. (3.16) we get

$$\tau_a = 2 + \frac{D - 1}{\gamma_{ar}}.$$  

(3.35)
Since $1 \leq D \leq 2$ we have the inequality

$$2 \leq \tau_a \leq \frac{5}{2}.$$  

(3.36)

One could say that the case $D = 2$ resembles the random neighbor model (where $A = S$) in the sense that the avalanche is all boundary but of course it makes no sense to talk about the radius of an avalanche in the random neighbor model.

### 3.3.7 Conjecture of the values of the critical exponents

Using $\gamma_{ra} = 1/2$ (compactness and isotropy) and $\gamma_{tr} = 4/3$ (self-avoiding random walk of activation fronts) and estimating $\tau_t = 15/7$ we can calculate all the other critical indices via the scaling relations Eq. (3.16) and Eq. (3.17). The result is displayed in Table 3.1, in which we also list the measured values. The data were averaged over $10^8$ avalanches in a $L = 200$ system in the measurements of $\tau_s$ and $\tau_t$. The other exponents are measured in a $L = 100$ system. Figure 3.3 shows all the measurements listed in Table 3.1.

<table>
<thead>
<tr>
<th>Exponent</th>
<th>Ref. [42]</th>
<th>Percolation</th>
<th>Measured $\pm 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tau_s$</td>
<td>$\frac{15}{7} \approx 2.14$</td>
<td>$\frac{187}{91} \approx 2.05$</td>
<td>$\frac{48}{23} \approx 2.09$</td>
</tr>
<tr>
<td>$\tau_t$</td>
<td>$\frac{14}{13} \approx 2.27$</td>
<td>$\frac{15}{13} \approx 2.14$</td>
<td>$2.14$</td>
</tr>
<tr>
<td>$\tau_r$</td>
<td>$\frac{1}{3} \approx 2.33$</td>
<td>$\frac{401}{180} \approx 2.10$</td>
<td>$\frac{41}{24} \approx 2.19$</td>
</tr>
<tr>
<td>$\gamma_{sr}$</td>
<td>$\frac{13}{6} \approx 2.17$</td>
<td>$\frac{187}{91} \approx 2.05$</td>
<td>$\frac{41}{24} \approx 2.10$</td>
</tr>
<tr>
<td>$\gamma_{st}$</td>
<td>$\frac{3}{2} \approx 1.87$</td>
<td>$\frac{11}{6} \approx 1.64$</td>
<td>$1.63$</td>
</tr>
<tr>
<td>$\gamma_{sr}$</td>
<td>$\frac{3}{1} \approx 2.33$</td>
<td>$\frac{91}{48} \approx 1.90$</td>
<td>$\frac{41}{24} \approx 2.19$</td>
</tr>
<tr>
<td>$\gamma_{sa}$</td>
<td>$\frac{1}{2} \approx 1.17$</td>
<td>$\frac{48}{25} \approx 1.10$</td>
<td>$1.06$</td>
</tr>
<tr>
<td>$\gamma_{tr}$</td>
<td>$\frac{5}{3} \approx 1.25$</td>
<td>$\frac{1}{1} \approx 1.33$</td>
<td>$1.36$</td>
</tr>
<tr>
<td>$\gamma_{ta}$</td>
<td>$\frac{15}{8} \approx 0.63$</td>
<td>$\frac{48}{25} \approx 0.53$</td>
<td>$\frac{1}{1} = 0.50$</td>
</tr>
</tbody>
</table>

Table 3.1: The critical exponents defined in Eq. (3.13) and Eq. (3.14) in the undirected abelian sandpile model. The first column is the result of Majumdar and Dhar based on the assertion that $\tau_a = 13/6$ and $\gamma_{tr} = 5/4$ [42]. The second column shows the values estimated from the scaling relations along with the percolation results in 2-d: $\tau_s = 187/91$, $\gamma_{sr} = 91/48$, and $\gamma_{sa} = 1$. The values in the third column rely on $\gamma_{tr} = 4/3$ which is based on the argument by Zhang [43] and the ansatz $\tau_t = 15/7$. Finally, we list the measured exponents in a $L = 100$ lattice, see Figure 3.3.
3.3. SCALING ARGUMENTS

Figure 3.3: Simulation results in a 2-dimensional system with BTW relaxation rules. The slope of the straight line in the log$_{10}$ − log$_{10}$ plot determines the critical exponent. All the results are listed in Table 3.1.

The agreement with the conjectured values is excellent. In Table 1 we have also listed the values one would obtain with the analytical result for percolation clusters in a 2-dimensional lattice at the percolation threshold. The clusters in a 2-d lattice at the percolation threshold are fractal with an exponent $91/48$ which is to be identified with the exponent $\gamma_{sr}$. The exponent $\tau_s$ describing the percolation cluster size distribution equals $187/91$ in two dimensions, and since multiple topplings never occur (by definition) $S = A$. The exponents with a reference to time $T$ are not defined.
3.4 Summary

The introduction of the neighborhood relations given by an underlying lattice will induce spatial correlations as well as turn the problem into transport problem. In a Bethe lattice the BTW relaxation rules given by Eq. (3.1) introduces exponentially decreasing correlations while a hypercubic lattice give rise to a power-law decreasing correlations.

Furthermore, the average size of avalanches will scale with system size. This is a consequence of the transport problem where the rate of flow into the system has to be transported across the boundary, which is the only place dissipation takes place. We observe power-law behavior of the distribution function of avalanche sizes. Heuristic arguments related to the transport properties of the model are given to indicate the origin of the power-law behavior.

Finally, we have derived scaling relations between the host of critical exponents. Additional constraints – (a) the compactness along with the isotropy of avalanches, (b) a relation between the avalanche size and the number of multiple topplings at the origin of the avalanche, and (c) the self-avoiding random walk performed by the activation fronts – are derived. All in all, we have $N-1$ equations to determine the $N$ exponents ($N = 16$). Based on simulations we fix one exponent and calculate the remaining exponents according to the relations derived. Thus, we conjecture that the power-law exponent corresponding to the distribution of avalanche sizes equals $48/23 \approx 2.09$ which is to be compared with our measurements of $2.08(5)$. 
Chapter 4

Nonconservative models

4.1 A laymans geophysics

The structure of the earth is a shell structure. The outer layer of the earth is called the crust and it is on the average 35 km thick beneath the continents while it is 7–8 km thick beneath the oceans. The following layer is the mantle which is usually subdivided into the upper mantle and the lower mantle going down to 670 km and 2891 km in depth, respectively. Beneath the mantle is the core: an outer fluid core from 2891 km down to 5150 km depth and from there down to 6371 km a solid inner core, see Figure 4.1.

![Diagram of the earth's structure](image)

Figure 4.1: The structure of the earth. The schizosphere going down to 10–20 km depth and the lithosphere which is about 100 km in depth are not shown.

Earthquake data contain lots of information about the structure of the earth. It was on the basis of earthquake data that Beno Gutenberg in 1912 delineated the
boundary between the mantle and the (outer) core [44]. Also, it was records of a 1929 earthquake in New Zealand which enabled Inge Lehman in 1936 to show that there was a solid inner core within the liquid outer core [45].

In 1967 it was discovered that the outermost shell of the earth going down to about 100 km – the lithosphere – consists of a number of rigid plates flowing on a viscous medium denoted the asthenosphere, see e.g. [46]. The lithosphere, which consists of the crust and part of the upper mantle, is so cold that the heat is transported by conduction. The asthenosphere, however, is warm enough to transport heat by convection in which the relevant time scale is million of years.

4.2 The earthquake mechanism

The convective flow in the asthenosphere causes the plates to move relatively to one another causing strain to build up at the boundary of the plates. The lithosphere is so rigid that the strain is released very sudden through earthquakes. Thus the worldwide occurrence of earthquakes outline the plate boundaries, see Figure 4.2.

\[\text{Figure 4.2: The world-wide occurrence of shallow earthquakes.}\]

With increasing temperature the plastic deformation dominates more and more. At a critical temperature the plastic deformation takes over the role of earthquakes as the main process of strain release. In fact, most of the earthquakes occur in the upper part of the lithosphere (see Figure 4.3) named the schizosphere which extends to about 10 – 20 km in depth where the temperature has reached about 300° C which is the critical temperature of continental rock types [47].
4.2. THE EARTHQUAKE MECHANISM

Earthquakes do occur with a hypocenter (focus) deeper than 100 km but approximately 75% occur in the lithosphere. There has never been recorded an earthquake with a hypocenter within the lower mantle where any strain is released by plastic deformation.

The plate boundaries are divided into three main classes: the constructive (or divergent) boundary, the destructive (or convergent) boundary, and finally a conservative boundary. Plate material is created, destroyed, and conserved, respectively [49]. We will dwell on the latter plate boundary which will serve to illustrate a simple mechanical model of earthquake occurrence.

The San Andreas fault in California, U.S.A., where the Pacific Plate and the North American Plate make up the boundary, is an example of a conservative boundary which is called a transform fault. The boundaries move parallel to each other with a rate of approximately 4 cm/year. When the fault cannot sustain the strain any more an earthquake is nucleated, see Figure 4.4.

The rupture propagates with a velocity around 4 km/s which is only a little smaller than the velocity of the primary P-waves which is from 6 – 8 km/s. The duration of an earthquake varies from fractions of a second (very small earthquakes) to minutes (very large earthquakes) so the extension of a rupture is from very small length scales to say 1000 km. The relative displacement of one side of the fault with respect to the other is of the order of centimeters to meters. For example, in the 1906 San Francisco earthquake in the San Andreas fault the relative displacement was in the range 0.6 – 6 m.

![Figure 4.3: The occurrence of earthquakes as a function of depth. The data are from Imperial Valley in the period 1977 – 1983 with a total of 865 earthquakes [48].](image)
4.2.1 Measurements of earthquakes

Until 1935 the size of an earthquake was measured by the damage they cause, which is a rather unreliable and subjective measure. C. F. Richter introduced in 1935 the concept of earthquake magnitude as a measure of the size of an earthquake.

An earthquake causes elastic waves to propagate through the interior of the earth (body waves) or on the surface of the earth (surface waves). The earthquake magnitude is a measure of the amplitude of the elastic waves. For example, the surface wave magnitude $M_S$ is defined by

$$M_S = \log_{10}(\frac{A}{T}) + q(\Delta, h) + a,$$

where $A$ is the amplitude in micrometers of the waves with periodicity $T$ seconds, $q$ is a phenomenological function correcting the decrease in amplitude caused by the distance $\Delta$ (measured in degrees) between the seismometer and the epicenter (the projection of the hypocenter (focus) to the surface of the earth) and the focal depth $h$ in kilometers. $a$ is an empirical constant [49]. Using the surface waves with a period of $T = 17 - 23$ sec. we find that

$$M_S = \log_{10} A_{20} + 1.66 \log_{10} \Delta + 2.0$$

when $h < 50$ km.
4.3. SELF-SIMILARITIES IN EARTHQUAKES

The earthquake magnitude was widely used as a measure of the size of an earthquake until 1977 where it was replaced by the seismic moment $M_0$ defined as

$$M_0 = \int_{\Omega} \mu u \, d\Omega.$$  \hfill (4.3)

$\Omega$ denotes the fault area, $\mu$ is the shear modulus (a measure of the resistance of a body to shearing strain), and $u$ being the relative displacement of one side of the fault to the other. If the shear modulus is a constant we find that

$$M_0 = \mu \Omega \frac{1}{\Omega} \int_{\Omega} u \, d\Omega = \mu \Omega \langle u \rangle,$$  \hfill (4.4)

where $\langle u \rangle$ is the average displacement of the fault. The advantages of the seismic moment to the surface wave magnitude $M_S$ are numerous. The surface wave magnitude saturates for very large earthquakes ($M_S \approx 7-8$) which is not the case for the seismic moment. Also, the error in the measurements of the surface wave magnitude is much larger than the corresponding error in the estimates of the seismic moment.

4.3 Self-similarities in earthquakes

4.3.1 The energy-frequency relation of earthquake occurrence

In 1944 B. Gutenberg and C. F. Richter discovered that the rate of occurrence of earthquakes with magnitude $M$ greater than $m$ is given by the relation

$$\log_{10} N(M > m) = a - bm,$$  \hfill (4.5)

where $a$ and $b$ are constants for a given fault [3]. Measurement of the parameter $b$ yields a wide range of values in different faults. Values of $b$ from 0.80 to 1.06 for small earthquakes and 1.23 to 1.54 for large earthquakes have been recorded [50].

The energy $E$ (proportional to the seismic moment) released during the earthquake increases exponentially with the earthquake magnitude, that is,

$$\log_{10} E = c + dm,$$  \hfill (4.6)

where the parameter $d$ is 1 and $\frac{3}{2}$ for small and large earthquakes, respectively [51]. By use of Eq. (4.5) and Eq. (4.6) we obtain a power law for the number of observed earthquakes with an energy release greater than $E$:

$$N(E_0 > E) \sim E^{-\frac{d}{2}} = E^{-B}.$$  \hfill (4.7)

Note that $B$ is in the same range for both small and large earthquakes, namely, $0.80 - 1.05$, see Figure 1.1(a).
4.3.2 Spatiotemporal behavior of earthquakes

Earthquakes display a complex spatiotemporal behavior in addition to the self-similarity in the statistics of the earthquakes. The underlying fault structure displays a fractal structure, where the number of faults as a function of fault length exhibits a power-law behavior. Furthermore, the earthquakes display spatiotemporal correlations. The spatial distribution of epicenters is fractal. There are short and long-range temporal correlations between nearby earthquakes. Short-range correlations are expressed by the Omori law which states that the number of earthquakes after a large earthquake decreases inversely with the elapsed time after the main event, see Figure 1.1(b) [4]. Study of long-term correlations between large earthquakes is subject to uncertainties because of the limited time of observation along with the rareness of large earthquakes. However, there are indications that long-range correlations exists between large earthquakes [5].

The temporal complexity is obviously of dynamical origin but also the statistics as well as the geometrical fractal structure displayed by the faults and by the spatial distribution of epicenters should be explained as a result of a dynamical process.

4.4 Earthquake modeling

The fractal behavior of earthquakes is not a consequence of a given geometrical structure like the fractal structure in the fault system. We cannot assume that a spatial fractal structure is preexisting. Nor is the lack of a characteristic scale a consequence of a random process. The phenomena calls for a more complete dynamical description in which both the geometrical fractal structure of the fault system as well as the additional scaling laws emerge naturally. We take the view that the seismic system evolves (self-organizes) into a dynamical state which is an attractor of the dynamics. The self-organization is a complex dynamical process as is the dynamics on the attractor.

Though the dynamics of earthquakes is very complex there are some simple basic components which have to enter a model of earthquakes: (a) earthquakes are generated by a very slow continuous drive of a fault, (b) the occurrence of earthquakes is intermittent, i.e., they occur as abrupt rupture events when the fault can no longer sustain the stress, and (c) there are two time scales involved in the process; one is related to the stress accumulation while the other, which is orders of magnitude smaller, is associated to the duration of the abrupt releases of stress.

There are different approaches when modeling the dynamics of earthquakes. One approach is to model the complete geological system and drive it uniformly and
observe the resulting spatiotemporal structure.

A more modest and successful approach is to discuss the internal dynamics of a single fault. A simplified model of the internal dynamics of a single fault including the basic components mentioned above is suggested by R. Burridge and L. Knopoff [52]. The one-dimensional version discussed by Burridge and Knopoff was taken up and explored by J. M. Carlson and J. S. Langer [18]. H. Nakanishi studied the model as a map (cellular automata) [53, 54] and very recently G. L. Vasconcelos, M. de S. Vieira, and S. R. Nagel discussed a similar model [55]. The model was further investigated in two dimensions by M. Otsuka [56].

4.4.1 A two-dimensional spring-block model

The two-dimensional model contains a large set of variables but it is possible to transfer the model into a rather simple map containing all the basic components listed above.

Since it is highly instructive we derive the mapping induced by this model: We consider a two-dimensional version of the Burridge-Knopoff model where the fault is represented by a two-dimensional network of blocks interconnected by springs. Each block is connected to the four nearest neighbors. Additionally, each block is connected to a single rigid driving plate by another set of springs as well as connected frictionally to a fixed rigid plate, see Figure 4.5.

![Figure 4.5: The geometry of the spring-block model. The two-dimensional system of blocks connected by springs. The force on the blocks increases uniformly as a response to the relative movement of the rigid plates.](image)

The blocks are driven by the relative movement of the two rigid plates. When
the force on one of the blocks is larger than some threshold value $F_{th}$ (the maximal static friction), the block slips. We assume that the moving block slips to the zero force position. Slip of one block will redefine the forces on its nearest neighbors. This may lead to instabilities of the neighboring blocks and thus, as a result, in further slips and a chain reaction (earthquake) can evolve. The total number of slips following a single initial slip event is a measure of the size (seismic moment) of the earthquake.

We begin by mapping the two-dimensional spring-block model into a continuous cellular automaton model. We define an $L \times L$ array of blocks by $(i,j)$, where $i,j$ are integers, $1 \leq i,j \leq L$. The displacement of each block from its relaxed position on the lattice is defined as $x_{i,j}$. The total force exerted by the springs on a given block $(i,j)$ is expressed by

$$F_{i,j} = K_1 \cdot [2x_{i,j} - x_{i-1,j} - x_{i+1,j}] + K_2 \cdot [2x_{i,j} - x_{i,j-1} - x_{i,j+1}] + K_L \cdot x_{i,j},$$

(4.8)

where $K_1$, $K_2$, and $K_L$ denotes the elastic constants, see Figure 4.6.

![Figure 4.6: A detailed picture of a given block $(i,j)$ and its surroundings.]

When the two rigid plates move relative to each other, the total force on each block increases uniformly (with a rate proportional to $K_L \cdot V$, where $V$ is the relative velocity between the two rigid plates) until one site reaches the threshold value and the process of relaxation begins (an earthquake is triggered). It can easily be shown
that the redistribution of strain after a local slip at the position \((i,j)\) is given by the relation

\[
F_{i\pm 1,j} \rightarrow F_{i\pm 1,j} + \delta F_{i\pm 1,j}, \\
F_{i,j\pm 1} \rightarrow F_{i,j\pm 1} + \delta F_{i,j\pm 1}, \\
F_{i,j} \rightarrow 0,
\]

(4.9)

where the increase in the force on nearest-neighbor blocks is

\[
\delta F_{i\pm 1,j} = \frac{K_1}{2K_1 + 2K_2 + K_L}F_{i,j} = \alpha_1 F_{i,j}, \\
\delta F_{i,j\pm 1} = \frac{K_2}{2K_1 + 2K_2 + K_L}F_{i,j} = \alpha_2 F_{i,j},
\]

(4.10)

For simplicity, we denote the elastic ratios by \(\alpha_1\) and \(\alpha_2\), respectively. Note that this relaxation rule is very similar to the BTW model [11]. However, when \(K_L > 0\), the redistribution of the force is nonconservative. In the context of the spring-block model, \(K_L > 0\); otherwise no driving force is possible. Thus, the spring-block model is nonconservative in nature with respect to the redistribution of force. The level of conservation is \(2\alpha_1 + 2\alpha_2\). However, with respect to energy (e.g., elastic energy) the model is nonconservative even if the redistribution of force is conservative.

Additional differences between the spring-block model and the old BTW model should be noticed. (1) The strain on the critical site is set to zero when relaxed. (2) The redistribution of strain to the neighbors is proportional to the strain in the relaxing site. (3) If \(K_1 \neq K_2\), \(\alpha_1 \neq \alpha_2\), this model is also anisotropic. In the following we restrict the discussion to the isotropic case \(K_1 = K_2 = K\) (\(\alpha_1 = \alpha_2 = \alpha\)) and the relaxation rules is given by

\[
F_{nn} \rightarrow F_{nn} + \alpha F_{i,j}, \\
F_{i,j} \rightarrow 0,
\]

(4.11)

where \(F_{nn}\) denote the nearest-neighbor blocks of the relaxing block \((i,j)\) and \(\alpha = K/(4K + K_L)\). We will refer to \(4\alpha\) as the (bulk)level of conservation.

The rules for the driving of the model are motivated by the dynamics of earthquakes in which there are two separate time scales. One is defined by the motion of the tectonic plates, and the other is the duration of an earthquake. The former time scale is much larger than the latter. We separate the time scales by considering the earthquake as instantaneous, that is, we do not drive the system during an earthquake. The algorithm for simulating the system is the following: Define random initial strains in the system. Strain is accumulated uniformly across the system as
the rigid plates move. When the strain in a certain site is above the threshold value $F_{th}$ this site will relax according to Eq. (4.11). This may cause neighboring sites to exceed the threshold value, in which case these sites relax simultaneously, and so on. The triggered earthquake will stop when there are no sites left with a strain above the threshold value. Strain starts to accumulate once again.

We continue this process to get proper statistics of the occurrence of earthquakes. The total number of slips in an earthquake is proportional to the change in total force (which we have check numerically) which in turn is a measure of the seismic moment (energy released). Consequently, we define the energy released during an earthquake as the total number of relaxation involved in the event.

In Figure 4.7 we plot the results of a simulation with $\alpha = 0.20$ for different system sizes. The conclusion drawn from the graphs is that the dynamics of the system is scale invariant; it displays SOC behavior.

Figure 4.7: The probability density of having an earthquake of energy $E$ as a function of $E$ for $\alpha = 0.20$. The different curves refer to different system sizes $L = 15, 25, 35, 50$. The cutoff in energy distribution scales with system size $L$.

This is surprising. The knowledge gained from the branching process seems to support the conjecture that the introduction of a nonconservative relaxation rule will leave the system subcritical [16]: a finite correlation length which does not scale with system size will appear. However, we must bear in mind that branching processes only deal with systems without spatial correlations. Thus we cannot apply the results of branching processes to systems where spatial correlations may occur. Chapter 5 and 6 are devoted to further investigations of the two-dimensional spring-block model defined above, but in the remaining part of this chapter we focus on the origin of criticality in nonconservative systems.
4.4.2 Coupled oscillators

To gain some insight into the behavior of such systems we note that the dynamics of a single degree of freedom, i.e., one block, is periodic. The strain is induced at a uniform rate and is released whenever the strain reaches the critical value, see Figure 4.8.

![Figure 4.8: The strain as a function of time when the system consists of only one single block. The behavior is equivalent to that of a single oscillator.](image)

The two-dimensional spring-block model can be regarded as a set of locally coupled oscillators where the avalanche dynamics constitutes a nonlinear coupling. The model is very similar to systems of coupled biological oscillators such as neurons. In order to see the possible characteristic behavior of the latter models we introduce the following globally coupled model of oscillators which is motivated by neural oscillators and by the spring-block model.

We discuss $N$ identical oscillators with a constant drive $V$. Let $F_i$ denote the phase of the $i$’th oscillator. The oscillators fire/collapse when they reach a critical value $F_{th}$. A collapsing oscillator transfers an amount of $\alpha F_{th}/N$ to the other oscillators, but like a neuron, will not receive a perturbation after a collapse. I.e., if the $i$’th oscillator collapses

$$ F_j \rightarrow F_j + \frac{\alpha F_{th}}{N}, $$

$$ F_i \rightarrow 0, \quad (4.12) $$

where the $j$’th oscillator has not yet collapsed during the ongoing avalanche. A very similar model is presented by R. E. Mirollo and S. H. Strogatz [57].

It is easily shown that the characteristic behavior of this model is a global synchronization of the phases. If two oscillators synchronize they cannot de-synchronize.
since both the coupling and the driving are global. Assume that the system has two clusters 1 and 2 of synchronized oscillators. Let \( N_1 \) and \( N_2 \) denote the number of oscillators in cluster 1 and 2, respectively. The distance between the phases of the two clusters will decrease if \( N_1 > N_2 \). When the cluster with size \( N_1 \) collapses an amount of \( \frac{N_1 \alpha F_{th}}{N} \) is transferred to every oscillator in cluster 2, and when the cluster with size \( N_2 \) collapses an amount of \( \frac{N_2 \alpha F_{th}}{N} \) is transferred to every oscillator in cluster 1. After one “cycle” the distance between the phases is reduced by \( \frac{(N_1 - N_2) \alpha F_{th}}{N} \). When the distance becomes smaller than say \( \frac{N_1 \alpha F_{th}}{N} \) the collapse of cluster 1 synchronizes all the oscillators.

Initially, the synchronization is due to the fluctuations in the local density of the phases (strains). If the initial values are chosen randomly between 0 and 1 the synchronization will generally occur when \( \alpha F_{th} > \frac{1}{N} \). Thus global coupled oscillators will synchronize even for a very weak (nonconservative) interaction.

In Figure 4.9 we present a graph of the average size of clusters as a function of time in a system with \( N = 1000 \) globally coupled oscillators and \( F_{th} = 4 \). The different curves refer to different values of the parameter \( \alpha \). The coupling between the oscillators decreases with decreasing \( \alpha \).

![Figure 4.9: The average number of synchronized oscillators as a function of time for \( \alpha = 0.8, 0.6, 0.4, \) and 0.2. The elapsed time before total synchronization occurs increases the weaker the coupling between the oscillators. The state of totally synchronized oscillators is an absorbing state. Once the oscillators synchronize they will stay synchronized forever.](image)

A similar behavior is common in biological systems. We mention only a few examples of global synchronization: the synchronization of fireflies flashing, the pacemaker cells in the heart, chirping of crickets, and neural synchronization in the
visual system [57].

However, no initial synchronization occurs if the interaction is too weak ($\alpha F_{th} < 1/N$). The system will stay in a state where the phases of the different oscillators move independently. All the “avalanches” are of size one; the system is in this sense localized. The two attractors – the globally synchronized state and the localized state – are quite trivial in their behavior, even if the dynamical route to synchronization may be complicated.

When the oscillators are coupled locally there might be other possibilities in between the two extreme limits. To study the locally coupled oscillators we introduce the following two-dimensional model: The $i$’th oscillator is assumed only to couple to its nearest neighbors. If $F_{i,j}$ exceeds a threshold value $F_{th}$ then

$$F_{nn} \to F_{nn} + 1, \quad (4.13)$$
$$F_{i,j} \to 0. \quad (4.14)$$

This model was introduced by H. J. S. Feder and J. Feder but in a slightly different context [58].

We present in Figure 4.10 the average cluster size of totally synchronized oscillators versus time. The system evolves into a synchronized state in which the phases are distributed within a few clusters.

![Figure 4.10: (a) The average number of synchronized sites in the system of locally coupled oscillators with size $L = 35$ and threshold value $F_{th} = 4$. This particular model does not display total synchronization, but the phases are totally synchronized within the few clusters. (b) The corresponding avalanche distribution when the system has reached the final partially-synchronized state.](image-url)

Synchronization occurs even with local coupling. Because the interaction between the oscillators is large they synchronize into several cluster. Synchronization
occurs only in nonconservative systems. However, this attractor is not robust to noise. It is destroyed even by a very small amount of noise. The dramatic effect of noise is easily observed. We alter the relaxation rule slightly by adding a small random number in the range of 0 to 0.001 to the collapsing oscillator. The system no longer synchronizes into a state with a few single clusters, and if we measure the avalanche distribution the system displays SOC. The results are presented in Figure 4.11.

Figure 4.11: The Feder model ($L = 35$ and $F_{th} = 4$) with dynamical noise. (a) The average number of totally synchronized oscillators. (b) The distribution of avalanche sizes. It is a power-law distribution.

Thus we reach the following conclusion: Coupled oscillators can display different behavior. Between the trivial behavior of independent uncorrelated oscillators and completely synchronized oscillators additional, more complicated attractors exist. Though this may not be the only possibility, we believe, that the latter attractor is to be identified with the dynamical attractor in nonconservative systems displaying SOC. The tendency to global synchronization is destroyed by any amount of noise and the resulting dynamics is only partially synchronized. In other words, this model is critical because correlations develop between the strains on the neighboring blocks. This has a very strong influence on the spatiotemporal behavior as we will discuss in chapter 5.

4.4.3 Global versus local drive

Using the results from the previous section we come to realize two basic points about nonconservative models. First, the uniform perturbation is a crucial element in driving nonconservative models to criticality. However, we also found that introduction
of noise moved the synchronized locally coupled model to the SOC state. Thus it is very interesting to study the effect of nonconservation in lattice models with random perturbation. In the following we restrict the discussion to 2-dimensional square lattices.

To move continuously between the limit of continuous (uniform) and random driving we perturb the system randomly by adding a fixed amount $\delta F$ to a randomly chosen site, that is, $F_{i,j} \rightarrow F_{i,j} + \delta F$. If the value at any site exceeds the threshold value $F_{i,j} \geq F_{th}$, the system relaxes according to

$$
F_{nn} \rightarrow F_{nn} + \alpha F_{i,j},
$$

$$
F_{i,j} \rightarrow 0.
$$

(4.15)

Note that the random drive represents an effective noise proportional to $(\delta F)^{1/2}$. The limit of small $\delta F$ is the limit of continuous drive.

The system is conservative when $\alpha = 1/4$. The dissipation occurs only at the boundary where $C_i < 4$. Y.-C. Zhang [43] and L. Pietronero et al. [38] discuss the conservative model in which $\delta F$ is chosen uniformly in the interval [0,1]. If $\alpha < 1/4$ the system is nonconservative: dissipation occurs in the bulk of the system as well as at the boundary.

We focus on the nonconservative model with $\alpha = 0.20$. Figure 4.12 shows the distribution function of avalanche sizes $P(S = s)$ for different choices of $\delta F$ which is fixed during the simulation. We use $\delta F = 1, 1/4, 1/16, 1/32, 1/64, 1/128$ and observe a qualitative change in the form of the distribution function.

The avalanches are localized for “large” values of $\delta F$. The distribution function is essentially an exponentially decreasing function. No scaling is observed when we change the system size. For “small” values of $\delta F$ the distribution function approaches a power-law behavior. Moreover, the distribution function scales with system size implying that the avalanches are no more localized.

A phase transition from a localized system into a critical system occurs between Figure 4.12(d) and (e). The noise (which is proportional to $\delta F^{1/2}$) is no longer large enough to destroy the local synchronization of the lattice. The correlation length grows until it reaches the system size. Obviously, noise plays two contradicting roles in this model. The noise destroys the possibility of global synchronization, but the noise does not necessarily destroy the correlations caused by the tendency of the system to synchronize. Any amount of noise will prevent global synchronization while the noise must be of a “minimal size” to destroy the correlations.
4.5 Summary

We have introduced the concept of plate tectonic motion which serves as a basis for the introduction of the two-dimensional spring-block model of a simple fault. We show that the quasi-static two-dimensional spring-block model is equivalent to an extended, nonconservative system, where the level of conservation is a function of the elastic parameters in the model. The distribution of energy released during an earthquakes exhibits a power-law behavior; hence the model is critical despite the inherent nonconservative nature.

We address the problem of the origin of the observed critical behavior in the nonconservative model. The behavior of a single block corresponds to that of a single oscillator. A system of uniformly driven, globally coupled oscillators displays total synchronization when the interaction between the oscillators is not too weak. A system of uniformly driven, locally coupled oscillators displays “nearly-global” synchronization, but this behavior is not robust to noise. A small amount of noise destroys the nearly-global synchronization, but the correlation survives and the system is capable of exhibiting long-range correlations although the system is nonconservative. If the noise is exceptionally large it destroys the correlations as well.

The effect of noise is clearly demonstrated in a nonconservative system with random local drive. The behavior of this model shows a phase transition into criticality when the effective noise (which is brought about by the driving) becomes smaller than a given finite value.
Figure 4.12: The nonconservative model with $\alpha = 0.20$ and $L = 25, 50$ when (a) $\delta F = 1$, (b) $\delta F = 1/4$, (c) $\delta F = 1/16$, (d) $\delta F = 1/32$, (e) $\delta F = 1/64$, and (f) $\delta F = 1/128$. 
Chapter 5

The spring-block model

5.1 Self-organization

In the remaining part of this tale we will discuss the intriguing behavior of the uniformly driven nonconservative spring-block model with the relaxation rules defined by Eq. (4.11). Note that the model is deterministic. The randomness in the model is inherited from the initial random state of the system. Clearly a self-organization process has to occur before long range synchronization/correlation can develop. Otherwise, avalanches would stay completely localized in a model with any amount of nonconservation like in the branching process. The running average of earthquake sizes illustrates the self-organization procedure, see Figure 5.1. The initial configuration is a random uncorrelated configuration. The average grows gradually, indicating the auto-correlation of the system into the critical state. Since the system auto-correlates slowly the maximal sizes of avalanches grow slowly. This growth is limited only by the system size which defines the avalanche size cutoff.

On the other hand if we start with a system in the critical state the average will immediately stabilize into the system size dependent value. This is also shown in Figure 5.1.

The avalanches are generated by correlated clusters of sites, which in turn are generated by the avalanches. The correlated clusters are modified by interactions with other clusters through avalanches, however they do not completely lose their identity. We will later discuss the dynamical implications of this.

A change in the parameters of the models will not destroy the self-organization process, but will modify it strongly. That is the origin for our results. A more detailed picture of this organization can be easily derived in one dimension, where the boundary between clusters is a point and not a line.
5.2 Distribution function, scaling, and critical indices

Let \( P(E, L) \) be the distribution function of having an avalanche of size \( E \) in a system of linear size \( L \). If the distribution function is a power law, we define the power-law exponent \( B \):

\[
P(E, L) \sim E^{-(1+B)}.
\]

(5.1)

The only constraint in this model is that the blocks have to move with an average velocity \( V \). Thus the geometrical size of the largest avalanches can at most scale with the area of the system. However, unlike the conservative case this does not supply us with any constraint about the shape or the scaling of the avalanche distribution. The distribution function can be localized, extended, or a power law with any exponent. A localized regime should be observed at least for very small \( \alpha \) where sites will move independently like in a system of uncoupled oscillators.

Simulations of the spring-block model show, that the continuous, nonconservative cellular automaton model exhibits SOC behavior for a wide range of \( \alpha \) values. The power-law exponent \( B \) depends on \( \alpha \). The dependence is shown in Figure 5.2.

When \( \alpha = 0 \) the movement of the blocks will become completely independent due to the lack of interaction. Therefore, we expect to see a transition to a localized
5.2. DISTRIBUTION FUNCTION, SCALING, AND CRITICAL INDICES

Figure 5.2: (a) Simulation results for a $35 \times 35$ system. Different curves refer to different levels of conservation. The slope of the curves become steeper as the $\alpha$ value is decreased. The graphs correspond to $\alpha = 0.25, 0.20, 0.15, 0.10, 0.075,$ and $0.025,$ respectively. Note that the last point is not critical. (b) The power-law exponent $B$ as a function of the elastic parameter $\alpha$. Below $\alpha = 0.05$ there is a transition to exponential decay. The arrows indicate the actual measured $B$ values for earthquakes [50].

behavior, characterized by a change from a power-law dependence with a cutoff that depends on system size to an exponentially decreasing distribution function. This indeed occurs in this model when $\alpha \approx 0.05$. Note that for this value only 20% of the value at the critical site is redistributed.

There is another phase transition in the system when the slope of the power-law distribution function reaches the value 3, that is when the power-law exponent $B$ reaches the value of 2.0 ($\alpha \approx 0.07$, see Figure 5.2(b)). At this value the variance of the avalanches becomes well defined.

The model have a very wide range of exponents, between 0.22 ($\alpha = 0.25$) in the conservative case and 2.5 ($\alpha \approx 0.05$) in the transition point where the distribution function changes from a power law into an exponentially decreasing function.

In addition if the elastic constants are comparable ($K_1 \approx K_2 \approx K_L$) then the characteristic value is $\alpha \approx 0.20$. Therefore, we expect to see the empirically observed $B$ values in the neighborhood of $\alpha \approx 0.20$. The measured range of $B$ values for earthquakes [50] are indicated by arrow in Figure 5.2(b). They are indeed in this region. Thus the result, apart from providing an explanation of the observed power laws, also give an explanation of the observed variability in the Gutenberg-Richter $b$-values. One should not look for universal $b$ ($B$) values in nature.
5.2.1 Finite-size scaling

It is important to investigate how the finite size of the system effects the properties we measure. The trademark of SOC is the existence of a power-law distribution function of the avalanche sizes that scale with the system size. The scaling properties of the system are investigated by finite-size scaling analysis. That is, we make the ansatz that the probability density scales with system size as

\[ P(E, L) = L^{-\beta} \cdot g\left(\frac{E}{L^\nu}\right), \tag{5.2} \]

where \( g \) is a so called scaling function, and \( \beta \) and \( \nu \) are critical indices describing the scaling of the distribution function. The critical index \( \nu \) expresses how the finite-size cutoff scales with system size while the critical index \( \beta \) is related to the normalization (or rather renormalization) of the distribution function.

If the distribution function is a power law we have already shown that

\[ 1 + B = \frac{\beta}{\nu}, \tag{5.3} \]

see Eq. (3.9). We measured the three exponents \( B, \nu, \) and \( \beta \) systematically in systems with different values of \( \alpha \). We present in Figure 5.3 two examples of finite-size scaling in a system with \( \alpha \) equal to 0.25, and 0.20, respectively. These graphs show that finite-size scaling works well. This verifies the criticality of the model as well as the absence of any characteristic length scale associated with nonconservation.

![Figure 5.3: Finite-size scaling analysis for systems of sizes \( L = 25, 35, \) and 45. (a) A conservative system, \( \alpha = 0.25 \). The scaling indices \( \nu = 3.3 \) and \( \beta = 4.2 \). (b) A nonconservative system with \( \alpha = 0.20 \) with \( \nu = 2.2 \) and \( \beta = 4.25 \). Additional measurements are listed in Table 5.1, see page 60.](image-url)
Figure 5.4 shows the results for the dependence of the critical exponents $\nu$ and $\beta$ on $\alpha$ in the region $0.10 \leq \alpha \leq 0.25$. The exponent $B$ changes continuously in the range $0.05 \leq \alpha \leq 0.25$. However, in the scaling indices we see a very sharp transition when going from the conservative into the nonconservative regime. The scaling index $\nu$ drops from 3.3 to 1.8 when changing $\alpha$ from 0.25 to 0.245.

![Graph showing critical indices $\nu$ and $\beta$ as functions of conservation.](image)

Figure 5.4: The critical indices $\nu$ (filled symbols) and $\beta$ (open symbols) as functions of the conservation. Notice the sharp transition when a small nonconservation is introduced.

We attribute this change of scaling to a transition in the temporal behavior of the avalanches. Introduce the integrated amount of local activity in the avalanche as a third dimension. When the system is conservative the avalanches have the shape of a cone, where the height is approximately of the same scale as the radius. However, when the system is nonconservative even the smallest dissipation level would define a maximal number of relaxation at a given site. Therefore the avalanches will become flat. This implies a change in the scaling behavior of the avalanches from 3 to 2. We believe that the width of the transition will depend on the system size. Another interesting feature in the nonconservative region of the model is that though the exponent $B$ changes strongly, $\nu$ is almost a constant. Still, the relationship between the exponents Eq. (5.3) is fulfilled in all the measured region.

### 5.2.2 Boundary conditions

The boundary is an integrated part of any finite system. The model would not be well-defined without a specification of the boundary conditions. The data we presented in the last section implies that there is a dependence of the exponents on the parameters of the model. It is also well known that existence of criticality
Table 5.1: The critical exponents in the nonconservative spring-block model as a function of $\alpha$. The scaling relation, Eq. (5.3) is fulfilled within numerical accuracy (except for $\alpha = 0.22$). For comparison we also list the critical exponents of the original BTW model.

<table>
<thead>
<tr>
<th>Model</th>
<th>$1 + B$</th>
<th>$\nu(\pm0.10)$</th>
<th>$\beta(\pm0.10)$</th>
<th>$\beta/\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.25$</td>
<td>1.22 $\pm$ 0.05</td>
<td>3.3</td>
<td>4.2</td>
<td>1.27 $\pm$ 0.05</td>
</tr>
<tr>
<td>$\alpha = 0.2499$</td>
<td>1.22 $\pm$ 0.05</td>
<td>3.1</td>
<td>3.95</td>
<td>1.27 $\pm$ 0.06</td>
</tr>
<tr>
<td>$\alpha = 0.245$</td>
<td>1.17 $\pm$ 0.05</td>
<td>1.7</td>
<td>2.2</td>
<td>1.29 $\pm$ 0.11</td>
</tr>
<tr>
<td>$\alpha = 0.24$</td>
<td>1.25 $\pm$ 0.05</td>
<td>1.8</td>
<td>2.6</td>
<td>1.44 $\pm$ 0.11</td>
</tr>
<tr>
<td>$\alpha = 0.22$</td>
<td>1.65 $\pm$ 0.05</td>
<td>2.0</td>
<td>3.7</td>
<td>1.85 $\pm$ 0.13</td>
</tr>
<tr>
<td>$\alpha = 0.20$</td>
<td>1.89 $\pm$ 0.10</td>
<td>2.2</td>
<td>4.25</td>
<td>1.93 $\pm$ 0.12</td>
</tr>
<tr>
<td>$\alpha = 0.175$</td>
<td>2.06 $\pm$ 0.10</td>
<td>2.35</td>
<td>4.8</td>
<td>2.04 $\pm$ 0.12</td>
</tr>
<tr>
<td>$\alpha = 0.15$</td>
<td>2.22 $\pm$ 0.10</td>
<td>2.35</td>
<td>5.15</td>
<td>2.19 $\pm$ 0.13</td>
</tr>
<tr>
<td>$\alpha = 0.10$</td>
<td>2.72 $\pm$ 0.10</td>
<td>2.3</td>
<td>6.6</td>
<td>2.87 $\pm$ 0.18</td>
</tr>
<tr>
<td>BTW model</td>
<td>1.1 $\pm$ 0.05</td>
<td>2.0</td>
<td>2.3</td>
<td>1.15 $\pm$ 0.08</td>
</tr>
</tbody>
</table>

depends on the boundary conditions. A model with BTW dynamics and reflecting boundary conditions (conservative), for example, cannot display any critical behavior. However, no systematic study of this issue was done even for the original BTW model.

In the spring-block models there is very large range of possible boundary conditions. We can control the properties of the boundary by changing the $\alpha$ at the boundary, henceforth called $\alpha_{bc}$.

For the spring-block problem there are two extreme possibilities: (a) The blocks in the boundary layer are connected only to blocks within the faults implying $\alpha_{bc} = \alpha/(1-\alpha)$, the boundary is free [22]. (b) The blocks in the boundary layer are coupled to an imaginary boundary block by springs, $\alpha_{bc} = \alpha$, the boundary is open. In all the simulations shown up till now we have used open boundary conditions. Note that the free boundary conditions is more conservative than the open boundary conditions. Neither the free nor the open boundary conditions is probably physical realistic, but the actual boundary conditions must be in between the two extreme limits. In the context of continuous cellular automata models one can even introduce a totally conservative boundary conditions, $\alpha_{bc} = 1/3$ which we will denote as a reflecting boundary.

First, we choose the physically interesting case of the free boundary conditions.
In Figure 5.5(b) we show the change of the exponent $B$ as the level of conservation is changed in the model with free boundary conditions and the open boundary condition.

![Figure 5.5](image)

Figure 5.5: (a) Simulation results for a $35 \times 35$ system with free boundary conditions. Different curves refer to different levels of conservation. The slope of the curves become steeper as the $\alpha$ value is decreased. The graphs correspond to $\alpha = 0.245, 0.20, 0.15, 0.10, 0.05$, and $0.01$, respectively. (b) The power-law exponent $B$ as a function of the elastic parameter $\alpha$. Filled symbols correspond to the model with open boundary conditions, while the open symbols are the measurements in a model with free boundary conditions.

Note that the free boundary conditions is totally reflective in the conservative model so the model cannot display criticality. The change in the boundary conditions induces a dramatic change in the exponents. The exponents in the system with the free boundary conditions are much lower than in the system with the open boundary conditions. Also, the localization transition in a system with the free boundary conditions is lowered to $\alpha \simeq 0.01$.

To check this dependence more carefully we changed the boundary conditions at the edge continuously from open boundary conditions $\alpha_{bc} = \alpha$ to reflecting boundary conditions $\alpha_{bc} = 1/3$. We see a continuous change of the exponent $B$ when going from the open boundary conditions, $\alpha_{bc} = 0.20$, to a value of $\alpha_{bc} \simeq 0.28$ which is larger than the $\alpha_{bc} = \alpha/(1 - \alpha) = 0.25$ used in free boundary conditions. Above this value the systems seem to be noncritical in the sense that no power laws are seen. The distribution function no longer scale with system size although the size of the maximal avalanches scale with system size. We have observed similar behavior in systems with other bulk-values of $\alpha$ but we did not make a detailed study.
5.3 Complex spatiotemporal behavior

The attractor of the nonconservative spring-block model is quite complex. We can explore the attractor by the temporal signatures of the model. Before going into the details we note that the system goes through a periodic process of strain accumulation. During the earthquakes a certain amount of the strain dissipates. This implies the existence of a cycle in the temporal signature. However, this does not imply complete periodicity which will occur only in the synchronized case. The existence of partially synchronized clusters implies partial periodicity.

To investigate the temporal signatures of the spring-block model we focus on the internal degrees of freedom as well as the the temporal sequence of earthquakes. Only the latter is observed in nature. We begin by discussing the fluctuations in the average strain per lattice site. We measure the temporal behavior of \( \langle F \rangle = \frac{1}{L^2} \sum_{i,j} F_{i,j} \), where the time is defined as the amount of stress accumulated per lattice site. Figure 5.6 shows the resulting graph for \( \alpha = 0.10 \) and 0.20.

![Figure 5.6](image)

Figure 5.6: The average strain per lattice site in a nonconservative system with size \( L = 50 \). (a) \( \alpha = 0.10 \). (b) \( \alpha = 0.20 \). The system with the lower value of \( \alpha \) is nearly periodic.

In order to investigate any underlying frequency in the temporal signal we present the power spectrum of the signals in the model with \( \alpha = 0.05, 0.10, \) and \( \alpha = 0.20 \). The inverse of the characteristic frequency depends on \( \alpha \) in a linear way, see Figure 5.7. Note that the conservative case is nonperiodic.

We note that unlike for the SOC in the conservative models [31] the fluctuation \( \Delta F_{\text{tot}}/L^2 \) does not decrease with system size. This is related to the fact that the avalanches are dissipative and scale with system size. The fluctuations will also scale with the system size, since they are generated by the avalanches.
Another measure of the internal correlations is provided by the temporal point-point correlation function

\[ \rho(\tau) = \frac{1}{L^2} \sum_{i,j} F_{i,j}(t)F_{i,j}(t+\tau) - \langle \frac{1}{L^2} \sum_{i,j} F_{i,j} \rangle_T^2 \]  

(5.4)

where \( \langle \cdot \rangle_T \) denotes a temporal average. We present the result for the conservative and nonconservative case in Figure 5.8.

The correlations in the conservative system decreases exponentially independent of system size. In the nonconservative model we observe two basic features. The first is an underlying periodicity related to the driving periodicity. The second is that the correlation decreases like a power law until some cutoff time where the periodicity is destroyed.
In nature earthquakes are the main observed feature in a seismic system. A very interesting aspect of earthquake is the correlation in the occurrence time. In Figure 5.9 we present two time sequences of earthquakes derived by our model, with $\alpha = 0.20$, of earthquakes with energy larger than 20 (small earthquakes) and 450 (large earthquakes), respectively.

Note that a quantitative definition of a “large earthquake” is a relative definition, since it depends on the system size. However, a qualitative definition of a large
earthquake would be, that the energy released during the earthquake is so large that the event is on the edge of the power-law distribution (i.e., where the distribution begins to show finite-size effects).

It is evident that the two time sequences are dramatically different. The sequence for the small earthquakes seems to be distributed randomly, while for the large earthquakes it is highly clustered. The centers of clustered earthquakes in the spring-block model are generally correlated strongly in space. The distribution function for the interoccurrence time $t$ (the time between earthquakes) with energy greater than $E$, $P_E(t = t_i)$ provides a measure of the temporal correlations. Another possible measure is given by the coefficient of variation. It is defined as the ratio of the square root of the variance of the interoccurrence time and the average of interoccurrence time:

$$C_V(E) = \frac{\sqrt{\text{Var}(t)}}{\text{exp}_E(t)}. \quad (5.5)$$

In a random signal the distribution function is simply an exponential function yielding $C_V(E) = 1$. In a periodic signal, $C_V(E) = 0$, while clustered earthquakes produce $C_V(E) > 1$.

We measured the coefficient of variation in the conservative model with $\alpha = 0.25$ and in a nonconservative model the characteristic value $\alpha = 0.20$ in the case of open boundary conditions. The results are presented in Figure 5.10.

Figure 5.10: The coefficient of variation $C_V(E)$ as a function of energy released during an earthquake. The results shown are from a $L = 35$ system with open boundary conditions. (a) The conservative case, $\alpha = 0.25$. (b) A nonconservative case, $\alpha = 0.20$. The large earthquakes are characterized by a coefficient of variation greater than 1, implying earthquake bunching.
No correlations are seen between earthquakes in the conservative model. Indeed, there seems to be some repulsion between earthquakes at intermediate energies ($C_V(E) < 1.0$). In the nonconservative model we see a clustering effect in large earthquakes, while random behavior is observed in small earthquakes. The decrease in the coefficient of variation for very large earthquakes is a finite-size effect and thus is related to the cutoff in the frequency-energy distribution.

Why do not small earthquakes show temporal clustering? The reason is, that contrary to large earthquakes, small earthquakes can not be correlated in time because they are not correlated in space. They cannot feel the presence of one another: a small event occurring in one part of the fault cannot influence the occurrence of a small event in a totally different part of the fault. This is, of course, not true for large earthquakes. However, if one does the same type of calculations for subsystems (subfaults), correlations between “smaller events” (where smaller refers to the total system) will appear. This implies that it might be very useful to pay attention to the much more numerous small events to get some statistical predictions for the few large events.

We also present the distribution function for interoccurrence time for small and large earthquakes in Figure 5.11. We see a complete change from a random exponential distribution function to a power-law decreasing distribution function.

![Figure 5.11: The probability distribution of interoccurrence time in a 35 × 35 system with open boundary conditions and with $\alpha = 0.20$. (a) We consider earthquakes with energy $\geq 20$. The probability distribution is an exponential distribution. (b) Results for earthquakes with $E \geq 250$. The distribution function is a power-law decreasing function.](image-url)
5.3. COMPLEX SPATIOTEMPORAL BEHAVIOR

The average interoccurrence time $\exp_E(t)$ scales as $E^B$. However, the characteristic time between larger earthquakes within a cluster is much smaller. It is simply the buildup time for the strain in the system which is independent on $E$.

The time between the clusters themselves is much larger than the average interoccurrence time. Both phenomena contribute to the rise in $C_V(E)$. The average interoccurrence time between large earthquakes is a measure of the buildup time for correlations in the system.

The same kind of temporal correlations is seen in real earthquakes. Small earthquakes seem to be uncorrelated; see Figure 6 of A. C. Johnston and S. J. Nava [7]. Large earthquakes display strong clustering; see Y. Y. Kagan and D. D. Jackson [5]. It might be very interesting to make the same kind of calculation for an earthquake catalog of some fault.

5.3.1 Multifractality in the temporal set of earthquake

In this section we discuss the temporal set of earthquakes which are generated by the deterministic dynamics of the spring-block model. We can characterize the occurrence of each avalanche by an occurrence time defined by the driving of the system (the increase of strain is proportional to the elapsed time).

We discuss the earthquake temporal sequence $S = \{E_{t_i}\}$, where $t_i$ is the occurrence time of the earthquake and $E$ is its energy. Subsets $S_{E_0} = \{E_{t_i} \in S | E \geq E_0\}$ of this temporal sequence are defined by assigning a minimal energy $E_0$ for the earthquake under consideration, see Figure 5.12.

![Figure 5.12: A subsequence $S_{150}$ of earthquakes. An earthquake with energy release $E$ occurring at time $t_i$ is represented by a delta function of height $E$.](image-url)
We can ascribe a fractal dimension $D(E_0)$ to each subset $S_{E_0}$, see \cite{2} for a detailed definition. We measure the number of intervals $N(\delta, E_0)$ of size $\delta$ needed to cover the subset $S_{E_0}$, and we find
\begin{equation}
N(\delta, E_0) \sim \delta^{-D(E_0)}.
\end{equation}
The fractal dimension $D(E_0)$ depends on the energy $E_0$. Figure 5.13(a) is the result of counting intervals for a system with size $L = 70$ when $\alpha = 0.20$ for three different energies $E_0 = 690, 1608,$ and $2297$. Note that the interval determining the fractal dimension grows with the minimal energy $E_0$. When the size of the system is enlarged the intervals belonging to the scaled avalanches grow like $L^\gamma$, $\gamma = 0.45 - 0.64$. Figure 5.13(b) displays the measured fractal dimension as a function of the minimal energy $E_0$.

\begin{figure}[h]
\centering
\subfloat[]{
\includegraphics[width=0.4\textwidth]{fig5.13a}
\caption{The number of intervals needed to cover the subset $S_{E_0}$ for $E_0 = 690, 1608,$ and $2297$. Notice the existence of a minimal temporal distance between two successive earthquake of a given minimal energy $E_0$. The range determining the fractal dimension grows with system size as well as with the minimal energy $E_0$.}
}
\subfloat[]{
\includegraphics[width=0.4\textwidth]{fig5.13b}
\caption{The fractal dimension $D(E_0)$ of $S_{E_0}$ as a function of the minimal energy $E_0$.}
}
\end{figure}

The fractal dimension for small earthquakes is 1, which is the characteristic of a random set. The results in Figure 5.13(b) are given only for energies where the interval is big enough to distinguish the fractal exponent. Nontrivial fractal dimension appears for larger earthquakes indicating clustering.

In order characterize the clustering more precisely we define a earthquake clustering function in the following way
\begin{equation}
g(t) = \langle n(t) \rangle_{t_i} - t \cdot \langle n \rangle,
\end{equation}
5.3. COMPLEX SPATIOTEMPORAL BEHAVIOR

where \( \langle n(t) \rangle_{t_i} \) is the number of avalanches in the interval \( (t_i, t_i + t) \) averaged over all \( t_i \) in \( S \), and \( \bar{\pi} \) is the average density of earthquakes. \( g(t) \) is simply related to an integral of a coarse grained correlation function. For a Poisson process \( g(t) \) is identically zero. We define \( g_{E_0}(t) \) as \( g(t) \) calculated for the subset \( S_{E_0} \), i.e., we restrict ourselves to the earthquakes with an energy release larger than \( E_0 \). The excess number of earthquakes a time \( t \) after the avalanche is \( dg(t)/dt \). For large earthquakes this behavior was reported to be a \( 1/t \) behavior.

To measure this correlation we generated large data sets. Since we expect to see different clustering for large and small earthquakes we measured the functions \( g_{E_0}(t) \) for the full range of energy. We give an example for the results in Figure 5.14.

![Figure 5.14: The earthquake clustering function \( g_{1608}(t) \) for a system of size \( L = 70 \) with \( \alpha = 0.20 \).](image)

The basic form of \( g_{E_0}(t) \) for the long-term behavior is

\[
g_{E_0}(t) \sim \begin{cases} \ln(t), & t < t_{co}(L); \\ f(E_0, L), & t > t_{co}(L). \end{cases}
\] (5.8)

This basic \( \ln \)-dependence does not depend on \( \alpha \) in the nonconservative critical models. This behavior seems to be universal. The exponent of the Omori law will be the same for all nonconservative models, namely \( -1.0 \). The temporal correlations of earthquakes in this model display universality even though the exponents for the frequency-energy distribution display no universality. The cutoff time \( t_{co}(L) \) for the \( \ln \)-behavior does not depend on the minimal energy of the set, but it depends on \( \alpha \) and the system size \( L \). We present results of \( f(E_0, L) \) when \( \alpha = 0.20 \) and \( L = 35 \) in Figure 5.15(a). Below a minimal energy too much fluctuations are seen to distinguish any correlation. Note that the drop at larger energies is a result of the fact that they are near the cutoff.
We study the scaling of $f(E_0, L)$ with the system size $L$. The finite-size scaling hypothesis

$$f(E_0, L) = L^{-\beta} g(E/L^\nu)$$

(5.9)

seem to work well with $\nu = 2.2$, which is the scaling exponent reported earlier, and $\beta = -1.6$, see Figure 5.15(b).

We further checked the dependence of the cutoff time $t_{co}$ on $\alpha$. For $\alpha = 0.25$ the cutoff time is zero, as the model becomes nonconservative a finite cutoff time appears. The cutoff increases with decreasing $\alpha$. For $\alpha = 0.20$ we checked the dependence of $t_{co}$ on system size $L$, and it grows algebraically with $L$: $t_{co}(L) \sim L^{0.7}$. Thus, for $\alpha < 0.25$ this cutoff time will diverge with system size. Note that the exponents describing the growth with system size of the scaling regions of phenomena related to the earthquake temporal sequence are small.

The conservative case is characterized by a complete lack of temporal correlations. This is a result of the conservative nature of the model. The avalanches are very big and multiple relaxations can occur at sites in the system during an avalanche. Since the avalanches are conservative the interoccurrence time is much smaller. Both effects drive the correlation time to zero. However, for nonconservative models the situation is very different. Since the model is nonconservative no multiple relaxations will occur during an avalanche. Moreover, such models can become critical only if internal correlations are created inside the lattice. Correlated clusters are created by the avalanches. One the other hand those clusters will be modified by avalanches but not completely destroyed. This is the reason for the
5.4. SUMMARY

An initial random system may self-organize into a state with long-range correlation. The self-organization process of the nonconservative spring-block system and hence the critical exponents depends on the level of conservation and the boundary conditions. In the critical nonconservative phase, the critical indices change continuously as a function of conservation. Furthermore, the exponents vary continuously when changing the boundary conditions smoothly. Consequently, there is no universality in the critical exponents.

The nonconservative nature creates, in addition to the variation of the exponents, the existence of nontrivial temporal correlation. We characterize the complex temporal structure of the earthquake activity. We discuss the correlations in the interoccurrence time of earthquakes. The model exhibits the features of real earthquake: the occurrence of small earthquakes is random, while the larger earthquakes seem to be bunched. The temporal sequence of the occurrence of earthquakes is observed to be multifractal, but the exponents governing the clustering are universal in nonconservative critical systems. The correlations disappear when the model becomes conservative.

The clustering observed in this model is within time scales larger than the buildup time of an earthquake. Thus the clustering we observe is a long-term clustering. The exact relationship between this and the Omori law for aftershocks (which might be related to instabilities generated by a large earthquake) is unclear. Similar calculations on earthquake data were done by Y. Y. Kagan and D. D. Jackson [5]. However, it is very hard to deduce any exact exponents from their data.
Chapter 6

1/f noise

It has been a long-standing puzzle why 1/f power spectra are seen in a variety of physical systems [10]. One of the ideas behind the BTW model was to explain the mechanism of 1/f noise, but the model does not contain a 1/f power spectrum. The spatiotemporal scaling in the self-organized critical state does not necessarily manifest itself in nontrivial exponents. H. J. Jensen et al. [41, 59] and later J. Kertész and L. B. Kiss [60] showed that the power spectrum of the activity was in fact 1/f^2, i.e., the spectrum of a random walk.

6.1 The weighted lifetime distribution

The proper relationship between the joint probability density \( P(S = s, T = t) \) of having an avalanche of size \( s \) and lifetime \( t \) and the power spectrum \( S(f) \) of linearly superimposed avalanches was worked out by H. J. Jensen et al. [59] and later generalized by K. Christensen et al. [41]. Introducing the \textit{weighted lifetime distribution}

\[
\Lambda(t) = \sum_s s^2 P(S = s, T = t),
\]

and assuming that \( \Lambda(t) \) exhibits a scaling behavior

\[
\Lambda(t) \sim t^\mu, \quad 0 \leq t_1 \leq t \leq t_2 < \infty,
\]

and is negligible outside this interval, they showed that the power spectrum scales like

\[
S(f) \sim \begin{cases} 
1, & \mu + 1 < 0 \\
\frac{f^{-(\mu+1)}}{f^{\alpha_\infty}}, & 0 < \mu + 1 < -\alpha_\infty \\
\frac{f^{\alpha_\infty}}{f^{\alpha_\infty}}, & \mu + 1 > -\alpha_\infty 
\end{cases}
\]

in the intermediate frequency regime \( 1/2 \pi t_2 << f << 1/2 \pi t_1 \).
Here, $\alpha_\infty$ is an exponent characterizing the shape of the avalanches and it can be proven $\leq -2$. When $f \to 0$ the power spectrum becomes white, since a linearly superimposed signal cannot contain temporal correlations beyond the longest possible lifetime of an avalanche. Thus, in order to get nontrivial exponents, $\mu + 1$ must be between 0 and 2, whereas in the BTW model $\mu + 1 = 3.1, 2.77, 2.69,$ and $2.56$ in dimension $2, 3, 4,$ and $5$, respectively [41]. Notice that the lower frequency cutoff scales inverse proportional with the upper temporal cutoff $t_2$ of the weighted lifetime distribution.

In the BTW model the local dynamical variable is conserved during the relaxations, whereas realistic models of natural phenomena, such as earthquakes, do not typically have any inherent conservation law. The spring-block models of earthquakes are very simple and robust nonconservative models. They can be viewed as generic “Ising” models of dissipative many-body systems. Also, the fact that one has a handle on the exponents through the level of conservation, suggest that a power spectrum with nonuniversal nontrivial exponents may emerge, as observed in nature.

6.2 Measurements in the spring-block model

6.2.1 The weighted lifetime distribution

We now focus on the weighted lifetime distribution for the avalanches defined in Eq. (6.1) and its relation to the power spectrum of linearly superimposed signals. The superimposed signals mimic the total activity in a very large, slowly driven system in which we can neglect the interference between different avalanches.

Figure 6.1 shows the weighted lifetime distribution $\Lambda(t)$ for various values of $\alpha$.

Indeed, they obey power laws with the exponent $\mu$ depending on the dissipation. The values of $\mu$ are in the range where nontrivial exponents for the power spectra are expected.
6.2. MEASUREMENTS IN THE SPRING-BLOCK MODEL

Figure 6.1: Distribution of weighted duration of avalanches in a system with size \( L = 100 \). (a) \( \alpha = 0.10 \), (b) \( \alpha = 0.15 \), and (c) \( \alpha = 0.20 \). The measured exponents \( \mu \) are \(-0.40\), \(-0.61\), and \(-0.92\), respectively.

6.2.2 The power spectrum

In order to measure directly the power spectrum of linearly superimposed avalanches we generate realizations of the number of relaxations per unit time step \( j(\tau) \) by repeating the following procedure:

1. Set \( j(\tau) = 0 \) for all \( \tau \).
2. Perturb the system until an avalanche is activated.
3. Choose a starting time \( \tau_0 \) at random, and increase \( j(\tau_0 + \tau) \) by the activity (number of relaxations per unit time step), for \( \tau = 1, \ldots, t \), where \( t \) is the lifetime of the avalanche.
4. Go to 2.

The assumption of no interference between different avalanches was essential for the derivation of Eq. (6.3). Time sequences \( j(\tau) \) generated according to the algorithm above, will, of course, fulfill this requirement.
For a given realization \( j(\tau) \) we calculate the Fourier transform \( \hat{j}(f) \), and define the power spectrum

\[
S(f) = |\hat{j}(f)|^2,
\]

which is a strongly fluctuating function of the frequency \( f \). We average over many different realizations (up to 1000 realizations) to reduce the variance in the power spectrum, which for one single realization is a 100% standard deviation. Figure 6.2(a)-(c) display the resulting power spectra measured by superimposing randomly the avalanches in a system with size \( L = 100 \).

![Figure 6.2](image-url)

Figure 6.2: The power spectrum of randomly superimposed avalanches with the same \( \alpha \) values as in Figure 6.1. The arrows indicate the upper frequency cutoff \( 1/2\pi \) for the scaling region. The exponents \( \varphi \) of the power spectrum are (a) \( \varphi = 0.50 \), (b) \( \varphi = 1.56 \), and (c) \( \varphi = 1.80 \). Within the numerical accuracy, those values are consistent with the values of \( \mu \). (d) The power spectrum of a system with size \( L = 250 \) driven continuously at a rate \( p = 0.001 \) with \( \alpha = 0.20 \). The straight line has a slope of \( -1.93 \).

We recall that the frequency region under consideration is \( 1/2\pi t_2 \ll f \ll 1/2\pi \), since \( t_1 = 1 \) by definition. The slopes of the straight lines are indeed in agreement with the predicted values obtained from substituting the values from Figure 6.1 into
Eq. (6.3). Table 6.1 lists all our simulation results. The exponents are roughly in the regime observed experimentally. Note the change in the sign of the slope $\mu$ is related to a change of the power-law exponent from values less than unity to values greater than unity. Alternatively, the time sequence $j(\tau)$ can be generated by a direct measurement of the activity in a slowly driven system of considerable size. Figure 6.2(d) presents the power spectrum of a system of size $L = 250$ driven with a finite rate $p = 0.001$ for $\alpha = 0.20$. The long-term correlations ignored by the random superposition method do not affect the shorter time scales of interest here. Indeed, the power spectrum seems to be in even better agreement with the exponent extracted from the exponent $\mu$ of the weighted lifetime distribution.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\mu + 1$</th>
<th>$\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.10$</td>
<td>$0.60 \pm 0.10$</td>
<td>$0.50 \pm 0.10$</td>
</tr>
<tr>
<td>$\alpha = 0.125$</td>
<td>$1.20 \pm 0.10$</td>
<td>$1.18 \pm 0.05$</td>
</tr>
<tr>
<td>$\alpha = 0.15$</td>
<td>$1.61 \pm 0.10$</td>
<td>$1.56 \pm 0.05$</td>
</tr>
<tr>
<td>$\alpha = 0.175$</td>
<td>$1.80 \pm 0.10$</td>
<td>$1.76 \pm 0.05$</td>
</tr>
<tr>
<td>$\alpha = 0.20$</td>
<td>$1.92 \pm 0.10$</td>
<td>$1.80 \pm 0.05$</td>
</tr>
<tr>
<td>$\alpha = 0.225$</td>
<td>$2.24 \pm 0.10$</td>
<td>$1.92 \pm 0.05$</td>
</tr>
<tr>
<td>2-d BTW model</td>
<td>$3.10 \pm 0.10$</td>
<td>$2.00 \pm 0.05$</td>
</tr>
</tbody>
</table>

Table 6.1: The measured exponents $\mu + 1$ and $\varphi$ for the weighted lifetime distribution and power spectrum, respectively. The exponents are, within numerical accuracy, consistent with Eq. (5.3). For $0.225 < \alpha \leq 0.25$ it was not possible to identify any scaling region. For comparison we also list the critical exponents for the original two-dimensional BTW model.

### 6.2.3 Scaling with system size

Finally, in order to verify that the interval, in which the power spectrum display $1/f$ noise, scales with system size, we generated the power spectrum for various system sizes, $L = 45, 70, 100$ with $\alpha = 0.20$. The power spectrum is shown in Figure 6.3. We observe that the lower frequency cutoff scales with system size $L$. This is a unique fingerprint of a many-body phenomenon. Hence, with this characteristic in mind, we urge that experiments be performed on systems of varying size in order to check the assertion that the $1/f$ noise is a critical dynamical many-body effect.
Figure 6.3: The power spectrum for systems with $\alpha = 0.20$ but different system sizes, $L = 45, 70, 100$. The lower frequency cutoff scales with system size, while the upper frequency cutoff is a constant. The exponent of the power spectrum seems to change slightly with system size.

### 6.3 Summary

Generic, deterministic, nonconservative models displaying self-organized criticality are shown to exhibit $1/f$ noise. The exponent of the power spectrum depends on the level of conservation.
Chapter 7

Acknowledgments

K. Christensen gratefully acknowledge the financial support of Carlsbergfondet, Augustinus Fonden, Løvens Kemiske Fabriks Forsknings Fond, Emil Herborgs Legat, Brookhaven National Laboratory, and the Danish Research Academy. This thesis would not bear my signature had it not been for the financial support from all these places.

I greatly appreciate the hospitality of Brookhaven National Laboratory where most of this work was accomplished, not on my own, but in collaboration with Dr. Zeev Olami and Senior scientist Per Bak. They have taught me all I know about SOC and wine.

During my stay at BNL I met a lot of people who make the world a better place to live: Klaus, Shobhana, Lie Ping, Michael, Wendy, Jim, Michael, Michael, and Sandra besides Zeev and Per are just a few that come to my mind.

Other physicists around the world, without whom I would not have been able to write this thesis: Hans C. Fogedby (who’s enthusiasm and insight into physics continue to amaze me), Henrik Flyvbjerg, and Mette Jørgensen (who is not a physicists but a linguist besides being a very dear friend), Denmark; Henrik J. Jensen, England; Kent B. Lauritzen, Germany; Iwan Jensen, U.S.A; Carmen Prado, Brazil; Y.-C. Zhang, Switzerland, and Hans Jacob S. Feder, Jens Feder, Paul Menkin, and Vidar Frette, Norway.

In fact, all members of the “Coorperative Phenomena Group” at the University in Oslo deserve to be mentioned, but a special thank-you to Ragnhild, who tempted me with “Gammel Dansk” (or was it the other way around) in order to make me attend the university early in the morning during the writing of this tale.
When travelling from one place to another it is very easy to feel the same way as Eponine in her song “On my Own” from the musical “Les Misérables”:

“And now I’m all alone again
Nowhere to turn, no one to go to
Without a home without a friend
Without a face to say hello to

And now the night is near, now I can make
Believe you’re here.

Sometimes I walk alone at night
When everybody else is sleeping.
I think of you and then I’m happy
With the company I’m keeping
The city goes to bed
And I can live inside my head.

On my Own ....”

I could mention a lot of people I would think of. My family in Denmark, my friends in Aarhus, specially the incomparable residents (present as well as former) in Koll. 7, 3., and Mogens, LU, Carsten, Lone, Thomas, Kjeld, Gurli, Ulla, and Steen (all of whom I had the pleasure of meeting in the States during my stay) not to forget Frede.

Marianne, Torben, and E.B. from Copenhagen, Denmark but also the Mørkøre Family in Oslo, Norway, would certainly come to my mind. I always thanks my lucky stars that I have crossed the path of all these wonderful people.
Appendix A


Z. Olami, H. J. S. Feder, and K. Christensen,

*Self-organized criticality*

in a

*continuous, nonconservative cellular automaton modeling earthquakes.*
Self-Organized Criticality in a Continuous, Nonconservative Cellular Automaton
Modeling Earthquakes

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(Received 19 August 1991)

We introduce a new nonconservative self-organized critical model. This model is equivalent to a quasistatic two-dimensional version of the Burridge-Knopoff spring-block model of earthquakes. Our model displays a robust power-law behavior. The exponent is not universal; rather it depends on the level of conservation. A dynamical phase transition from localized to nonlocalized behavior is seen as the level of conservation is increased. The model gives a good prediction of the Gutenberg-Richter law and an explanation to the variances in the observed $b$ values.

PACS numbers: 91.30.Px, 05.40.+j, 05.45.+b

The dynamics of earthquake faults may provide a physical realization of the recently proposed idea of self-organized criticality (SOC). Bak, Tang, and Wiesenfeld (BTW) introduced the concept of self-organized criticality: Dynamical many-body systems reach a critical state without the need to fine-tune the system parameters [1]. BTW showed that a certain class of systems drive themselves into a statistically stationary state characterized by spatial and temporal correlation functions exhibiting power-law behavior. Hence, the system has no intrinsic length or time scale and is in a sense critical. The study of the SOC systems has to a great extent been based on simulations using cellular automaton models. The majority of these simulations have been limited to conservative models. It has been suggested that the necessary (and sufficient) condition for SOC is indeed a conservation law [2,3]. This seems to be the situation for SOC models where perturbation is done locally as in the original BTW model [4]. Recently, though, it was shown that a special nonconservative model with a global perturbation displays SOC [5].

Earthquakes are probably the most relevant paradigm of self-organized criticality. In 1956 Gutenberg and Richter realized that the rate of occurrence of earthquakes of magnitude $M$ greater than $m$ is given by the relation

$$\log_{10} N(M > m) = a - bm.$$  \hspace{1cm} (1)

This is the Gutenberg-Richter law [6]. The parameter $b$ has been recorded to have a wide range of values for different faults. Findings of $b$ from 0.80 to 1.06 for small earthquakes and 1.23 to 1.54 for large earthquakes have been reported [7].

The energy (seismic moment) $E$ released during the earthquake is believed to increase exponentially with the earthquake magnitude,

$$\log_{10} E = c + dm,$$  \hspace{1cm} (2)

where the parameter $d$ is 1 and $\frac{1}{3}$ for small and large earthquakes, respectively [8]. Thus the Gutenberg-Richter law is transformed into a power law for the number of observed earthquakes with energy greater than $E$,

$$N(E_0 > E) \sim E^{-b/d} = E^{-b}.$$  \hspace{1cm} (3)

Note that $b$ is in the same range for both small and large earthquakes, namely, 0.80–1.05.

Bak and Tang indicated that the simple conservative SOC models can serve as a framework for explaining the power-law behavior, giving a $b$ value of 0.2 [9]. Similar results are obtained for two-dimensional models in [10,11]. Otsuka was the first to simulate a 2D version of the Burridge-Knopoff model and he found $b = 0.8$ [12]. Carlson and Langer proposed a 1D dynamical version of the Burridge-Knopoff model [13–15]. A similar quasistatic model in one dimension was investigated by Nakanishi [16,17].

We introduce a generalized, continuous, nonconservative cellular automaton model that displays SOC [18]. This model has several interesting aspects. First, it is directly mapped into a two-dimensional version of the famous Burridge-Knopoff spring-block model for earthquakes. Second, it displays a robust SOC behavior over a very wide range of conservation levels. Third, we find that the level of conservation has an impact on the power laws obtained. We see a transition from localized behavior into nonlocalized behavior as the level of conservation is increased. Finally, the dependence of the power laws on the conservation allows us to explain the wide variances in the Gutenberg-Richter law as a result of the variances of the elastic parameters.

The Burridge-Knopoff spring-block model is a two-dimensional dynamical system of blocks interconnected by springs. Each block is connected to the four nearest neighbors. Additionally, each block is connected to a single rigid driving plate by another set of springs as well as connected frictionally to a fixed rigid plate [see Fig. 1(a)]. The blocks are driven by the relative movement of the two rigid plates. When the force on one of the blocks is larger than some threshold value $F_{th}$ (the maximal static friction), the block slips. We assume that the moving block will slip to the zero-force position. This assumption is not essential for the behavior of the model as will become evident later on. The slip of one block will redefine the forces on its nearest neighbors. This can result in further slips and a chain reaction can evolve.

For the purpose of mapping the spring-block model into a cellular automaton model we define an $L \times L$ array of blocks by $(i,j)$, where $i,j$ are integers restricted to the
where the increases in the nearest-neighboring forces are
\[
\delta F_{i,j} \pm 1 = \frac{K_1}{2K_1 + 2K_2 + K_L} F_{i,j} = a_1 F_{i,j},
\]
\[
\delta F_{i,j} = \frac{K_2}{2K_1 + 2K_2 + K_L} F_{i,j} = a_2 F_{i,j}.
\]

For simplicity we denote the elastic ratios by $a_1$ and $a_2$, respectively. Notice that this relaxation rule is very similar to the well-known BTW model. However, when $K_L > 0$ the redistribution of the force is nonconservative. Thus, the well-established spring-block model, used to describe earthquakes, is nonconservative in nature. If $K_1 \neq K_2$ ($a_1 \neq a_2$) this model is also anisotropic. We will describe the phase diagram for this case in a forthcoming paper [19]. Furthermore, if $a_1 \neq 0$, $a_2 = 0$, we have a one-dimensional version of the spring-block model. Nakanishi [16,17] studied this model with the relaxation rule for the slipping block $i$ as a function of the excess force $F_i - F_{th}$, that is, $F_i \rightarrow \phi(F_i - F_{th})$. In our model $\phi = 0$ and the one-dimensional nonconservative systems are not critical.

In this paper we restrict ourselves to the isotropic case, $K_1 = K_2$ ($a_1 = a_2 = a$). The boundary condition of the model is rigid, implying that $F = 0$ on the boundary. The time interval between earthquakes is much larger than the actual duration of an earthquake. Thus, the mapping of the spring-block model into a continuous, nonconservative cellular automaton modeling earthquakes is described by the following algorithm.

(1) Initialize all sites to a random value between 0 and $F_{th}$.
(2) If any $F_{i,j} \geq F_{th}$ then redistribute the force on $F_{i,j}$ to its neighbors according to the rule
\[
F_{n,n} \rightarrow F_{n,n} + aF_{i,j},
\]
\[
F_{i,j} \rightarrow 0,
\]

where $F_{n,n}$ are the strains for the four-nearest neighbors. An earthquake is evolving.

(3) Repeat step 2 until the earthquake is fully evolved.
(4) Locate the block with the largest strain, $F_{max}$. Add $F_{th} - F_{max}$ to all sites (global perturbation) and return to step 2.

We measure the probability distribution of the size (the total number of relaxations) of the earthquakes, which is proportional to the energy released during an earthquake.

There are several differences between our model and the BTW model.

(i) The strain on the critical site is set to zero when relaxed. (ii) The redistribution of strain to the neighbors is proportional to the strain in the relaxing site [20]. (iii) The relaxation is not conservative. It will be conservative only when $K_L = 0$. If $K_L > 0$, the model will be nonconservative. In the context of the spring-block model,
$K_l > 0$; otherwise no driving force is possible. If we assume that all elastic constants are on the same scale ($K_1 = K_2 = K_l$) then $\alpha = 0.20$.

The continuous, nonconservative cellular automaton model exhibits SOC behavior for a wide range of $\alpha$ values. The exponent $B$ depends on $\alpha$. The dependence is shown in Figs. 2(a) and 2(b). To verify the criticality of the model we study the effect of increasing the system size $L$. We observe that for $\alpha =$const the exponent stays the same, while the cutoff in the energy distribution scales with system size. We give an example of this behavior in Fig. 3 which shows the results of simulations with $\alpha = 0.20$ for $L = 15, 25, 35, \text{and } 50$, giving $B \approx 0.91$.

By examining the scaling of the cutoff in the energy distribution as a function of the system size for $\alpha = 0.20$ we find that the cutoff scales with $L^{2.2}$ [19]. In the original BTW model the cutoff scales with the square of the system size $L$. This verifies the criticality of the model as well as the absence of any characteristic length scale associated with nonconservation. For the lower values of $\alpha$ ($\alpha \leq 0.10$) it is difficult to pursue this scaling since the large values of $B$ in this region make it very hard to obtain good statistics for the large events.

It is clear that if $\alpha = 0$ the movement of the blocks will become completely uncorrelated due to the lack of interaction. Therefore, we expect to see a transition to a localized behavior, characterized by a change from a power-law dependence to an exponential decay. This indeed occurs in this model for $\alpha \approx 0.05$. Notice that for this value only 20% of the value at the critical site is redistributed.

There is another transition in the slope of the critical exponent when $B$ reaches the value of 2.0 ($\alpha \approx 0.07$). For this value the variance of the avalanches becomes well defined.

Since other algorithms seem to be very dependent on external noise [5,11] we checked the effect of noise on our model. We introduced noise with zero mean and with a variance up to $0.25 E_{\text{th}}$ in each relaxing site for the case of $\alpha = 0.20$. This noise had no effect on the exponent and cutoff.

In conclusion we have shown that a continuous, nonconservative model can have a very robust SOC behavior. This is not in accord with the predictions based on differential equations [2,3]. Previous published models [9–11] neglect the presence of the overlying leaf spring, thus assuming $K_l = 0$. This implies that their models are conservative.

The model has a very wide range of exponents, between 0.22 for the conservative case and 2.5 for the transition point. In addition, if the elastic constants are comparable
(\(K_1 = K_2 = K_L\)) then the characteristic value is \(\lambda = 0.20\). Therefore, we expect to see the empirically observed \(B\) values in the neighborhood of \(\lambda = 0.20\). The measured range of \(B\) values for earthquakes [7] are indicated by the arrows in Fig. 2(b). They are indeed in this region.

Our model is extremely robust even under large noise. On the other hand, the model does not rely on the introduction of noise as is the case in Refs. [5,11]. This is also contradictory to previous models for self-organized criticality based on differential equations [2,3]. Our model includes the basic feature of the spring-block model: the existence of an enormous phase space of metastable states. The metastable states lose their stability only if they exceed the threshold value. The models based on driven diffusion equations do not share this feature. We believe this to be the fundamental property modeling earthquakes.

Another interesting feature of our model is the lack of universality. Changing the value of \(\lambda\) will alter the exponent but still keep the system critical.

While this model is obviously highly simplified we believe it provides a reasonable resemblance of the actual dynamical process associated with earthquake faults.

We would like to thank Per Bak for introducing us to this subject. We are also indebted to Jens Feder, Carmen Prado, Revach Zeev, and Henrik Flyvbjerg for valuable discussions. Z.O. thanks the Weizmann and Fulbright Foundations for support during this research. H.J.S.F. thanks NAFV for travel support. K.C. gratefully acknowledges the financial support of Carlsbergfondet, Augustinus Fonden, Løvens Kemiske Fabriks Forskningsfond, and the Danish Research Academy. All authors appreciate the support and hospitality of Brookhaven National Laboratory. This work was supported by the Division of Basic Energy Sciences, U.S. DOE under Contract No. DE-AC02-76CH00016.

[18] We use the term continuous cellular automaton to denote a system with continuous state variables. The concept of (discrete) cellular automaton is reserved for a system with discrete state variables.
[20] H. Takayasu and M. Matsuzaki, Phys. Lett. A 131, 244 (1988). Takayasu and Matsuzaki have considered a two-dimensional model which is quite similar to the Feder and Feder model [5]. The differences are that the amount of increase in the force on the neighboring blocks is a constant parameter of the model and that the force on the relaxed blocks cannot increase during an earthquake. We see no direct relationship between these two models and the two-dimensional Burridge-Knopoff spring-block model.
Appendix B

B.1 J. Geophys. Res. 97, 8729-8735 (1992)

K. Christensen and Z. Olami,

_Variation of the Gutenberg-Richter b values and nontrivial temporal correlations in a spring-block model for earthquakes._
Variation of the Gutenberg-Richter $b$ Values and Nontrivial Temporal Correlations in a Spring-Block Model for Earthquakes

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We show that a two-dimensional spring-block model for earthquakes is equivalent to a continuous, nonconservative cellular automaton model. The level of conservation is a function of the relevant elastic parameters describing the model. The model exhibits power law distributions for the energy released during an earthquake. The corresponding exponent is not universal. It is a function of the level of conservation. Thus the observed variations in the $b$ value in the Gutenberg-Richter law could be explained by a variation in the elastic parameters. We address the problem of the boundary conditions and display results for two extreme possibilities. Furthermore, we discuss the correlation in the interoccurrence time of earthquakes. The model exhibits the features of real earthquakes: the occurrence of small earthquakes is random, while the larger earthquakes seem to be bunched. Primarily, the results of our work indicate that the dynamic of earthquakes is intimately related to the nonconservative nature of the model, which gives birth to both the change in the exponent and the correlations in interoccurrence time.

1. INTRODUCTION

In the realm of experimental physics, physicists deal with the characteristic behavior of real physical systems. In the world where theoretical physics reigns, physicists are concerned with simplified models. The majority of these models are, of course, derived from real physical systems. The golden rule, when mapping a physical system into a model system, is to grasp only the important features of the relevant phenomena. Otherwise, the model system can very easily turn out to be too complex, so that it will be almost impossible to comprehend the mechanism, which is responsible for the observed behavior.

Likewise, some insight into the complicated dynamics of earthquakes may be derived from simplistic models that contain the essential features of earthquakes. Such a simple model, a spring-block model, was proposed by Burridge and Knopoff [1967]. Originally, the model was defined as a two-dimensional model, but Burridge and Knopoff had to restrict themselves to a one-dimensional version when performing experiments and simulations. Models made up of the same basic ingredients (still in one dimension) were analyzed by several authors [Carlson and Longer, 1989a,b; Nakanishi, 1990, 1991]. A two-dimensional version of the spring-block model was simulated by Otsuka [1972], who also suggested a quasi-static analysis of this model. Slightly different models were later innovated by Bak and Tang [1989], who suggested that the idea of self-organized criticality (SOC), which was first introduced in the context of complex dynamical systems, might apply to earthquakes. Similar models were suggested by Brown et al. [1991], Bunde and Brown [1991], and Ito and Matsuzaki [1990]. Different models were proposed by Feder and Feder [1991] and Olami et al. [1992].

The simplest test for these models is their ability to reproduce the Gutenberg-Richter law [Gutenberg and Richter, 1956] which states that the rate of occurrence of earthquakes of magnitude $M$ greater than $m$ is given by the relation

$$\log N(M > m) = a - bm,$$  

where $a$ and $b$ are constants for a given fault. Measurement of the parameter $b$ yields a wide range of values for different faults. Values of $b$ from 0.80 to 1.06 for small earthquakes and 1.23 to 1.54 for large earthquakes have been recorded [Pacheco et al., 1992]. The energy $E$ (proportional to the seismic moment) released during the earthquake increases exponentially with the earthquake magnitude,

$$\log E = c + dm,$$

where the parameter $d$ is 1 and 3/2 for small and large earthquakes, respectively [Eström and Dziewonski, 1988]. By use of (1) and (2) we obtain a power law for the number of observed earthquakes with energy greater than $E$

$$N(E_0 > E) \sim E^{-4} = E^{-B}. $$

Note that $B$ is in the same range for both small and large earthquakes, namely, 0.80$-$1.05.

While the models, simulated in the context of earthquake dynamics, produce power law for the frequency-energy distribution, they do not predict the proper $B$ nor can they explain the variability of the $B$ values.

Another important problem, associated with earthquake prediction, is the question of spatial and temporal correlation of earthquakes. It is observed that small earthquakes occur randomly in time. That is, the distribution of interoccurrence time of earthquakes with energy greater than $E$ is a Poisson distribution for small $E$ [Johnston and Nava, 1985]. Larger earthquakes, on the other hand, seem to be clustered [Kagan and Jackson, 1991]. Also, the clustered earthquakes are strongly correlated in space.

Though much effort has been addressed to correlations, especially for large earthquakes, we are not aware of any significant achievements. We believe that a more fundamental approach is required.

In this paper we map a two-dimensional version of the spring-block model for earthquakes into a continuous, nonconservative cellular automaton model. This model has several fascinating aspects. First, it exhibits a robust self-organized critical behavior over a very wide range of conservation levels. That is, we observe power law distribu-

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tions for the relevant physical quantities, where the cutoff scales with the size of the system. Second, we find that the level of conservation has an impact on the power laws obtained. The level of conservation is a function of the elastic constants in the spring-block model. Therefore, the wide variances in the Gutenberg-Richter law could be interpreted as a result of the variances of the relevant elastic parameters. Finally, the model exhibits a clustering in the occurrence of large earthquakes, while the small earthquakes are distributed randomly in time.

2. THE MODEL

Burridge and Knopoff [1967] simplified the problem by discussing the internal dynamics of one single fault rather than the dynamics of a complex system of coupled faults. We consider a two-dimensional version of their model where the fault is represented by a two-dimensional network of blocks interconnected by springs. Each block is connected to the four nearest neighbors. Additionally, each block is connected to a single rigid driving plate by another set of springs as well as connected frictionally to a fixed rigid plate (see Figure 1a). The blocks are driven by the relative movement of the two rigid plates. When the force on one of the blocks is larger than some threshold value \( F_{th} \) (the maximal static friction), the block slips. We assume that the moving block will slip to the zero force position. Slip of one block will redefine the forces on its nearest neighbors. This may lead to instabilities in the neighboring blocks and thus, as a result, in further slips and a chain reaction (earthquake) can evolve. The total number of slips following a single initial slip event is a measure of the size (seismic moment) of the earthquake.

As a first step, we map the two-dimensional spring-block model into a continuous cellular automaton model. We define an \( L \times L \) array of blocks by \((i,j)\), where \( i, j \) are integers, \( 1 \leq i, j \leq L \). The displacement of each block from its relaxed position on the lattice is defined as \( x_{i,j} \). The total force exerted by the springs on a given block \((i,j)\) is expressed by

\[
F_{i,j} = K_1 [2x_{i+1,j} - x_{i-1,j} - x_{i+1,j}] \\
+ K_2 [2x_{i,j} - x_{i-1,j} - x_{i+1,j}] \\
+ K_L x_{i,j},
\]

where \( K_1, K_2, \) and \( K_L \) denotes the elastic constants (see Figure 1b). When the two rigid plates move relative to each other, the total force on each block increases uniformly (with a rate proportional to \( K_L \cdot V \), where \( V \) is the relative velocity between the two rigid plates) until one site reaches the threshold value and the process of relaxation begins (an earthquake is triggered). It can easily be shown (see Appendix A) that the redistribution of strain after a local slip at the position \((i,j)\) is given by the relation

\[
F_{i\pm 1,j} \rightarrow F_{i\pm 1,j} + \delta F_{i\pm 1,j} \\
F_{i,j\pm 1} \rightarrow F_{i,j\pm 1} + \delta F_{i,j\pm 1} \\
F_{i,j} \rightarrow 0,
\]

where the increase in the force on nearest-neighbor blocks is

\[
\delta F_{i\pm 1,j} = \frac{K_1}{2K_1 + 2K_2 + K_L} F_{i,j} = \alpha_1 F_{i,j} \\
\delta F_{i,j\pm 1} = \frac{K_2}{2K_1 + 2K_2 + K_L} F_{i,j} = \alpha_2 F_{i,j}.
\]

For simplicity, we denote the elastic ratios by \( \alpha_1 \) and \( \alpha_2 \), respectively. Notice that this relaxation rule is very similar to the BTW model [Bak et al., 1987]. However, when \( K_L > 0 \), the redistribution of the force is nonconservative. In the context of the spring-block model, \( K_L > 0 \); otherwise no driving force is possible. Thus, the spring-block model is nonconservative in nature with respect to the redistribution of force. The level of conservation is \( 2\alpha_1 + 2\alpha_2 \). However, with respect to energy (e.g., elastic energy) the model is nonconservative even if the redistribution of force is conservative. Throughout this paper the terminology "conservative" and "nonconservative" refer to the redistribution of force.

Additional differences between our new model and the old BTW model should be noticed. (1) The strain on the critical site is set to zero when relaxed. (2) The redistribution of strain to the neighbors is proportional to the strain in the relaxing site. (3) If \( K_1 \neq K_2 (\alpha_1 \neq \alpha_2) \), this model is also anisotropic. In this paper we restrict the discussion to the isotropic case \( K_1 = K_2 (\alpha_1 = \alpha_2 = \alpha) \).

The definition of the boundary conditions is a very sub-
tle problem. There are two extreme possibilities: (1) The blocks in the boundary layer are connected only to blocks within the faults, the boundary is free. (2) The blocks in the boundary layer are coupled to a rigid boundary block by springs, the boundary is open. We refer to Appendix B for a detailed discussion of the boundary conditions.

Obviously, neither of the two choices is correct. It is very difficult to know what the proper boundary conditions are, and they are probably not the same for each fault. However, the actual boundary conditions must be somewhere between the two extreme limits given above. We will present results for both alternatives.

Brown et al. [1991] considered a model with a different geometry, \(1 \leq i \leq L_x, 1 \leq j \leq L_y, L_x \gg L_y\), where \(L_x\) is the slip direction. They used free boundary conditions in the slip direction and periodic boundary conditions in the perpendicular direction. The elastic constants were chosen to fulfill the conditions \(K_L \ll K_1 = K_2\). It is easy to see that this is an almost conservative model (e.g., \(K_L = 1, K_1 = K_2 = 25\) imply \(\alpha \approx 0.246\)). In this case the free boundary conditions is almost reflective, since the effective level of conservation in the edges is 0.98. We see no physical reason to assume that \(K_L \ll K_1, K_2\). A more reasonable assumption, is that all the elastic constants are in the same order of magnitude, \(K_L \approx K_1 \approx K_2\). This implies that the characteristic value of \(\alpha\) is 0.20. Rundle and Brown [1991] have investigated a square model with free boundary conditions for this particular value of \(\alpha\). It should be noticed that in both these models some randomness has to be introduced to avoid periodic behavior: the relaxing block does not slip to zero force position but rather to zero force position plus a random overshoot.

3. SIMULATION OF THE MODEL

The rules for the driving of our model are motivated by the dynamics of earthquakes. There are two time scales involved. One is defined by the motion of the tectonic plates, and the other is the actual duration of an earthquake. Since the first time scale is much larger than the second, we can separate the time scales. We consider the earthquake as instantaneous and do not drive the system during an avalanche. Thus the algorithm for simulating the system is the following: Define random initial strains in the system. Strain is accumulated uniformly across the system as the rigid plates move. When the strain in a certain site is above the threshold value \(F_{th}\), this site will relax according to (5) and (6). This may cause neighboring site to exceed the threshold value, in which case these sites relax simultaneously, and so on. The triggered earthquake will stop when there are no sites left with a strain above the threshold. Strain starts to accumulate once again.

We continue this process to get proper statistics of the total number of relaxations in the individual earthquakes. The total number of slips is proportional to the change in total force (which we have check numerically) which in turn is a measure of the seismic moment (energy released). Consequently, we define the energy released during an earthquake as the total number of relaxation involved in the event.

3.1. Power Law Exponents

The continuous, nonconservative cellular automaton model exhibits SOC behavior for a wide range of \(\alpha\) values for either boundary conditions. The exponent \(B\) depends on \(\alpha\) and on the chosen boundary conditions. In Figure 2 we show the change of the exponent \(B\) as the level of conservation is changed for the model with free boundary conditions. In Figure 3 we display the dependence for both kinds of boundary conditions. The model has a wide range of possible exponents. For the open boundary conditions, \(B\) is between 0.22 (\(\alpha = 0.25\)) and 2.5 (\(\alpha = 0.05\)). For the free boundary conditions the range is –0.08 (\(\alpha = 0.245\)) to 1.35 (\(\alpha = 0.01\)) (notice that at exact conservation the model with free boundary conditions is not well defined because the boundary is totally reflective). The model by Rundle and Brown [1991] with free boundary conditions and \(\alpha = 0.20\) gave \(B = 0.6\), which is in agreement with our results.

This variability could serve as an explanation for the variances in the observed \(B\). Note that the exponents for the
free boundary conditions are much lower than those of the open boundary conditions. This is a result of higher correlations between strains in the lattice. The relevant exponents seen for real earthquakes are near the characteristic value of $\alpha \approx 0.20$ for the open boundary conditions (see Figure 3).

We simulated the model on various lattice sizes $L$, using up to $5 \times 10^7$ avalanches to get accurate estimates of $B$. In Figure 4a we show results of simulations for the open boundary conditions with $\alpha = 0.20$ for $L = 15, 25, 35$, and $50$ giving $B \approx 0.91$. By examining the scaling of the cutoff in the energy distribution as a function of the system size (K. Christensen and Z. Olami, manuscript submitted to Physical Review A, 1992) we find that the cutoff scales with $L^{2.2}$; see Figure 4b. This verifies the criticality of the model as well as the absence of any characteristic length scale associated with nonconservation. Similar results, though with different $B$, are obtained for the free boundary conditions.

It is clear that if $\alpha = 0$, the movement of the blocks will become completely uncorrelated due to the lack of interaction. Therefore, we expect to see a transition to a localized behavior as $\alpha$ is decreased. This indeed happens for both kinds of boundary conditions but at surprisingly low $\alpha$. For the open boundary conditions this occurs at $\alpha \approx 0.05$. At this value a transition between a power law behavior and exponential decay is observed. For the free boundary conditions the transition emerges in a different manner. At $\alpha \approx 0.01$ the $B$ value is frozen and a length scale, a cutoff which is proportional to $\alpha$, appears. Notice that in this case only 4% of the strain is conserved. Those transitions are obviously not relevant for earthquakes but might be related to other systems.

3.2. Spatiotemporal Clustering of Large Earthquakes

A very interesting aspect of earthquake is the correlation in the occurrence time. In Figure 5 we present two time sequences of earthquakes derived by our model, with $\alpha = 0.20$, for earthquakes with energy larger than 20 (small earthquakes) and 450 (large earthquakes), respectively. Notice that a quantitative definition of a "large earthquake" is a relative definition, since it depends on the system size. However, a qualitative definition would be that the energy released during the earthquake is so large that the event is on the edge of the power law distribution (i.e., where the distribution begins to show finite size effects).

It is evident that the two time sequences are dramatically different. The sequence for the small earthquakes seems to be distributed randomly, while for the large earthquakes it is highly clustered. The centers of clustered earthquakes in our model are generally correlated strongly in space.

The distribution function for the interoccurrence time $t$ (the time between earthquakes) with energy greater than $E$, $P_E(t = t_i)$ provides a measure of those temporal correlations. Another possible measure is the coefficient of variation. It is defined as the ratio of the square root of the variance of the interoccurrence time and the average of interoccurrence time:

$$CV(E) = \frac{\sqrt{\text{var}(t)}}{\text{exp}_E(t)}$$  \hspace{1cm} (7)
For a random signal the distribution function is simply an exponential function yielding $C_V(E) = 1$. For a periodic signal, $C_V(E) = 0$, while clustered earthquakes will produce $C_V(E) > 1$.

We measured the coefficient of variation for the conservative model $\alpha = 0.25$ and for the characteristic value $\alpha = 0.20$ in the case of open boundary conditions. The results are presented in Figure 6.

No correlations are seen between earthquakes in the conservative model. Indeed, there seems to be some repulsion between earthquakes at intermediate energies ($C_V(E) < 1.0$). In the nonconservative model we see a clustering effect for large earthquakes, while random behavior is observed for small earthquakes. The decrease in the coefficient of variation for very large earthquakes is a finite size effect and thus is related to the cutoff in the frequency-energy distribution. Why do small earthquakes not show temporal cluster-

![Fig. 6. The coefficient of variation $C_V(E)$ defined in equation (7) as a function of energy released during an earthquake. The results shown are for a 35 x 35 lattice with open boundary conditions. (a) The conservative case, $\alpha = 0.25$. Notice that $C_V(E) \leq 1$ for all $E$. $C_V(E) = 1$ indicates that the occurrence of earthquakes is random. (b) A nonconservative case, $\alpha = 0.20$. For small earthquakes, $C_V(E) \leq 1$. The large earthquakes are characterized by a coefficient of variation greater than 1, implying earthquake bunching.](image)

![Fig. 7. The probability distribution of interoccurrence time for a 35 x 35 system with open boundary conditions and a level of conservation equal 0.80, $\alpha = 0.20$. (a) We consider earthquakes with energy $\geq 20$. The probability distribution is an exponential distribution. (b) Results for earthquakes with $E \geq 250$. The distribution function is a power law decaying function.](image)

The reason is that contrary to large earthquakes, small earthquakes cannot be correlated in time because they are not correlated in space. They can not feel the presence of one another: a small event occurring in one part of the fault can not influence the occurrence of a small event in a totally different part of the fault. This is, of course, not true for large earthquakes.

However, if one does the same type of calculations for sub-systems (subfaults), correlations between “smaller events” (where smaller refers to the total system) will appear. This implies that it might be very useful to pay attention to the much more numerous small events to get some statistical predictions for the few large events!

We also present the distribution function for interoccurrence time for small and large earthquakes in Figures 7a and 7b respectively. We see a complete change from a random exponential distribution function to a power law decaying distribution function.
The average interoccurrence time $\text{exp}_E(t)$ scales as $E^B$. However, the characteristic time between larger earthquakes within a cluster is much smaller. It is simply the buildup time for the strain in the system which is independent on $E$.

The time between the clusters themselves is much larger than the average interoccurrence time. Both phenomena contribute to the rise in $C_V(E)$. The average interoccurrence time between large earthquakes is a measure of the buildup time for correlations in the system.

The same kind of temporal correlations is seen in real earthquakes. Small earthquakes seem to be uncorrelated; see Figure 6 of Johnston and Nada [1985]. Large earthquakes display strong clustering; see Kagan and Jackson [1991]. It might be very interesting to make the same kind of calculation for an earthquake catalog of some fault.

These results are closely related to the fact that the model is nonconservative. The occurrence of large earthquakes in such a model is related to creation of correlated strains in the fault. Otherwise, the earthquakes would become completely localized. Since the correlations are not completely destroyed by intermediate shocks, clustering will occur with correlated centers. The nonconservative nature of the model creates correlations in the strain of a fault, which is an important feature in the creation of the relevant power laws and temporal correlation. The earthquakes are generated by correlated clusters of sites, which in turn are generated by the earthquakes themselves. The correlated sites are modified by interactions with other clusters through earthquakes.

For conservative models, however, large earthquakes involve a lot of activity inside the fault which completely destroy any correlations existing in the lattice. This explains why the earthquakes in a conservative model can not be correlated in time.

4. Conclusion

The results presented in this paper indicate very strongly that earthquakes are related to nonconservative SOC models. The nonconservative nature of the model is induced through the relaxation rules. The effects of this dissipative nature are quite profound. It creates a variation of the exponents and the existence of nontrivial temporal correlations in the system. Both features seem to be in accord with what is observed for actual earthquakes. This introduces a new set of ideas which appears to be associated with the problem of earthquake temporal and spatial correlations. We believe that further theoretical and experimental effort can clarify those intriguing problems.

As to the generality of the model we can only speculate. We believe that most nonconservative models which display criticality, i.e., power law behavior where the cutoffs scale with system size, are, at least qualitatively, similar to the model presented in this paper. The strong argument for this is that in order to create power law distribution functions in a nonconservative model, the model must be able to correlate in space. This will lead to temporal correlations as we have demonstrated in this model.

We may be able to improve our ability to estimate the probabilities of occurrence of large earthquakes, even with the limited information available about the history of earthquake occurrence and existing faults by utilizing the information hidden in the numerous small events!

**Appendix A**

Assume that the strain of a block at position $(i, j)$ is above the threshold value, that is,

$$F_{th} \leq F_{i,j} = K_1 (2z_{i,j} - z_{i-1,j} - z_{i+1,j}) + K_2 (2z_{i,j} - z_{i,j-1} - z_{i,j+1}) + K_L \cdot z_{i,j}.$$  \hfill (8)

If $\tilde{z}_{i,j}$ denotes the displacement of block $(i,j)$ from the relaxed position after the block has slipped to zero force position, then

$$0 = K_1 (2\tilde{z}_{i,j} - z_{i-1,j} - z_{i+1,j}) + K_2 (2\tilde{z}_{i,j} - z_{i,j-1} - z_{i,j+1}) + K_L \cdot \tilde{z}_{i,j}.$$  \hfill (9)

where we exploit the fact that nearest-neighbor blocks can not be supercritical at the same time, why

$$x_{i\pm 1,j} = \tilde{x}_{i\pm 1,j}$$
$$x_{i,j\pm 1} = \tilde{x}_{i,j\pm 1}.$$  \hfill (10)

The slipping block $(i, j)$ affects the strain on the nearest-neighbor blocks. As an example, we can calculate the change of force on block $(i, j+1)$. The force on block $(i, j+1)$ is

$$F_{i,j+1} = K_1 (2z_{i,j+1} - z_{i-1,j+1} - z_{i+1,j+1}) + K_2 (2z_{i,j+1} - z_{i,j} - z_{i,j+2}) + K_L \cdot z_{i,j+1}. \hfill (11)$$

Thus the change of force due to a slip at position $(i,j)$ is

$$\delta F_{i,j+1} = -K_2 \cdot dx_{i,j}.$$  \hfill (12)

Notice that the force on block $(i, j+1)$ may very well be affected by a slip at position $(i,j+2)$ but that does not interfere this argument. An expression for the change in displacement of block $(i,j)$,

$$dx_{i,j} = \tilde{x}_{i,j} - z_{i,j} \hfill (13)$$
is obtained by subtracting (8) from (9):

$$0 - F_{i,j} = [2K_1 + 2K_2 + K_L] \cdot dx_{i,j}. \hfill (14)$$

Finally, substituting (14) into (12), we find

$$\delta F_{i,j+1} = \frac{K_2}{2K_1 + 2K_2 + K_L} \cdot F_{i,j}. \hfill (15)$$

**Appendix B**

The boundary condition is free if the blocks in the boundary layer are connected only to blocks within the fault; that is, the force on a boundary block, say at site $(i,L)$, is given by

$$F_{i,L} = K_1 (2z_{i,L} - z_{i-1,L} - z_{i+1,L}) + K_2 (z_{i,L} - z_{i-1,L}) + K_L \cdot z_{i,L}. \hfill (16)$$

If this block slips, we find

$$0 - F_{i,L} = [2K_1 + K_2 + K_L] \cdot dx_{i,L}. \hfill (17)$$
resulting in
\[ \delta F_{i \pm 1, L} = \frac{K_1}{2K_1 + K_2 + K_L} \cdot F_{i, L} = \alpha_{bc} \cdot F_{i, L}. \] (18)

If the model is isotropic, \( K_1 = K_2 = K \), then
\[ \alpha = \frac{K}{4 \cdot K + K_L}. \] (19)

We can express the elastic ratio \( \alpha_{bc} \) used when boundary blocks slips in terms of \( \alpha \)
\[ \alpha_{bc} = \frac{\alpha}{1 - \alpha}. \] (20)

If a block in one of the corners slips, we use
\[ \tilde{\alpha}_{bc} = \frac{\alpha}{1 - 2 \cdot \alpha}. \] (21)

For a simulation of the model with open boundary conditions we use the same \( \alpha \) all over the lattice. That is, if block \( (i, L) \) slips we still increase the force on the three neighboring blocks \( (i \neq 1, L) \) with an amount equal to \( \alpha \cdot F_{i, L} \), where \( \alpha \) is defined in (6).

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Appendix C


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Scaling, phase transitions, and nonuniversality

in a

self-organized critical cellular-automaton model.
Scaling, phase transitions, and nonuniversality in a self-organized critical cellular-automaton model

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We present a two-dimensional continuous cellular automaton that is equivalent to a driven spring-block model. Both the conservation and the anisotropy in the model are controllable quantities. Above a critical level of conservation, the model exhibits self-organized criticality. The self-organization of this system and hence the critical exponents depend on the conservation and the boundary conditions. In the critical isotropic nonconservative phase, the exponents change continuously as a function of conservation. Furthermore, the exponents vary continuously when changing the boundary conditions smoothly. Consequently, there is no universality of the critical exponents. We discuss the relevance of this for earthquakes. Introducing anisotropy changes the scaling of the distribution function, but not the power-law exponent. We explore the phase diagram of this model. We find that at low conservation levels a localization transition occurs. We see two additional phase transitions. The first is seen when moving from the conservative into the nonconservative model. The second appears when passing from the anisotropic two-dimensional system to the purely one-dimensional system.

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I. INTRODUCTION

The common appearance of fractal structures in nature is a long-standing puzzle. How can systems that interact only locally create correlated structures with a great variety of length scales and scale invariance? We encounter this phenomenon in statistical physics when a phase transition takes place in an equilibrium system. The system exhibits fluctuations in all possible length scales, but the phase transition occurs only at a critical point, to which one has to fine-tune some external parameters of the system. Obviously, there is no reason to assume that natural systems are fine-tuned to a thermodynamic phase-transition point.

Recently, Bak, Tang, and Wiesenfeld (BTW) suggested that natural systems might organize themselves, without any fine-tuning, to a critical state through their local dynamics [1]. They named this kind of organization to criticality “self-organized criticality” (SOC). This behavior is usually characterized by a power-law distribution function of the activity (avalanches) in the critical dynamical system and by existence of scale invariance of the distribution function.

The idea of SOC went hand in hand with a definite cellular-automaton algorithm, suggested by BTW, initiating enormous scientific activity. Various different cellular automata were studied and a great effort was spent on deriving some theoretical understanding of the new models.

A common feature to most of the models was that the local dynamical rules obeyed a conservation law. There has been some question as to whether or not the conservation is a necessary requirement for SOC [2,3]. With regard to the generality of the concept of SOC, this is a critical aspect, since many natural phenomena have inherent nonconservative properties.

Simulation of a system, with local perturbations and nonconservation in the dynamical rules of the original BTW model, shows that a length scale is indeed introduced into the problem [4]. At the same time different models with some degree of local nonconservation, like the “game of life” and “forest fire model,” seem to be critical. However, it is very hard to follow the dynamics in detail in those models. In fact, the local dynamics can be “more than conservative,” so the average nonconservation in the latter models is not well defined; thus it is not clear to what extent the SOC is associated with the requirement of conservation.

Recently, Feder and Feder introduced a two-dimensional model (hereafter called the Feder model) with low level of nonconservation that seems to display criticality [5]. While this result is indeed very intriguing, the model suffers from several deficiencies. One is that without dynamical noise the model behaves periodically, showing no criticality. Even with noise, where the model exhibits criticality, the model behaves almost periodically. Also, the amount of nonconservation is not a fully controllable quantity, and it is somewhat unclear what the role of the nonconservation is.

Another general question concerns the universality of the models. An attempt to connect the problem of SOC to randomly driven Langevin equations was made by several authors [2,3]. The basic equations were directed nonlinear diffusion equations driven by random noise. The primary predictions of those models are that any kind of nonconservation will immediately destroy criticality and that a limited set of universal exponents will characterize the systems. It is doubtful whether the models based on differential equations have any connection to SOC due to the different nature of the dynamical rules. Moreover, the result of Feder and Feder [5] question the application of differential equations to this kind of prob-
lem. Still, it is quite interesting to see if the predictions based on nonlinear differential equations have any relevance to critical cellular-automata models.

We must notice that fractal behavior in nature usually does not display universality and in most relevant phenomena a great variety of exponents are seen. Two simple examples are the power-law distributions of energy released during an earthquake, which will be discussed later (Sec. IV), and the power-law distributions of \("1/f\) noise." No universal power-law exponent characterize either case.

If there exist any universality classes in the cellular-automata models, there are obviously a large number of them. It seems that almost any modification of the algorithms introduces different exponents [6,7]. Even small modifications of the original BTW model that leave all the symmetries the same, for example, can change the scaling and the exponents [8]. This might be related to the observed variance in the critical exponents.

In this paper we introduce a generalized continuous cellular-automaton model where the nonconservation is a controlled quantity [9,10]. This model has several fascinating aspects.

First, it can be directly mapped into the Burridge-Knopoff spring-block model of earthquakes [11]. Hence the relevant variables can be interpreted as the forces in a spring-block system and the nonconservation is simply defined by a ratio between the elastic constants. Though the motivation for the model is derived from the Burridge-Knopoff spring-block model, it can be regarded as a generic representation of a nonconservative system.

Second, this model displays SOC over a very wide range of conservation levels. That is, we observe power-law distributions for the relevant physical quantities, where the cutoff scales with the size of the system. Furthermore, we find that the level of conservation has an impact on the power laws and the scaling exponents of this system. In particular, this has implications for the power-law distributions of earthquakes.

Third, two phase transitions are observed for this model. The first transition, where the scaling of the avalanches changes discontinuously, occurs when going from the conservative into the nonconservative case. A second phase transition takes place in low conservation levels, where we observe a localization of the avalanches. Between the two phase transitions the exponents change continuously.

Fourth, we find that the isotropic version of our model displays a continuous transition from two-dimensional to one-dimensional scaling. But the impact on the power law is negligible.

Finally, we find that a change of boundary conditions has an effect on the criticality and the exponents. For two natural choices of boundary conditions of the Burridge-Knopoff model we find large differences between the exponents. The boundary of the system has a strong influence on the self-organization process; consequently it affects the critical exponents.

In Sec. II we describe how the model is derived from the Burridge-Knopoff spring-block model and discuss the difference between our model and related models. We define the critical power-law exponent, and the critical scaling indices are introduced in the concept of finite-size-scaling analysis. In Sec. III we present a detailed study of the exponents and the finite-size scaling of the isotropic model in two dimensions. We observe several phase transitions and a variability of the scaling exponents inside the critical phases. We present results for the free boundary conditions of the spring-block model, and some observations on the variability of the laws as a function of the boundary conditions. In Sec. IV we consider the relationship of our results to earthquakes. Finally, we discuss the anisotropic model and the complete phase diagram of this model in Sec. V. In conclusion, we interpret our results in a wider context.

II. THE CELLULAR-AUTOMATON MODEL, EXPONENTS, AND FINITE-SIZE SCALING

A. Derivation of the model

Our model can be considered as a general nonconservative cellular automaton; hence our results have general implications. However, it is important to notice that it can be related directly to a driven model of earthquakes. Therefore, we begin by deriving this model from the simplified model suggested by Burridge and Knopoff for the internal dynamics of one single fault.

We consider a two-dimensional version of their model where the fault is represented by a two-dimensional network of blocks interconnected by springs. Each block is connected to the four nearest neighbors. Additionally, each block is connected to a single rigid driving plate by another set of springs, as well as connected frictionally to a fixed rigid plate [see Fig. 1(a)]. The blocks are driven by the relative movement of the two rigid plates. When the force on one of the blocks is larger than some threshold value \(F_{\text{th}}\) (the maximal static friction), the block slips. We assume that the moving block will slip to the zero-force position. The slip of one block will redefine the forces on its nearest neighbors. This can result in further slips and a chain reaction (an avalanche) can evolve.

We define an \(L \times L\) array of blocks by \((i,j)\) where \(i,j\) are integers \(1 \leq i,j \leq L\). The displacement of each block from its relaxed position on the lattice is defined as \(x_{ij}\). The total force exerted by the springs on a given block \((i,j)\) is expressed by

\[
F_{ij} = K_1[2x_{ij} - x_{i-1,j} - x_{i+1,j}] + K_2[2x_{ij} - x_{i,j-1} - x_{i,j+1}] + K_L x_{ij},
\]

(1)

where \(K_1\), \(K_2\), and \(K_L\) denote the elastic constants [see Fig. 1(b)]. When the two rigid plates move relative to each other, the total force on each block increases uniformly (with a rate proportional to \(K_L V\), where \(V\) is the relative velocity between the two rigid plates) until one site reaches the threshold value and the process of relaxation begins (an earthquake is triggered). It can easily be shown (see Appendix A) that the redistribution of forces after a local slip at the position \((i,j)\) is given by the relation
Some differences between our model and other models should be noticed.

(1) The force on the critical site is set to zero when relaxed. The same rule is used by Feder and Feder, while in the BTW model, a constant amount (usually $F_{th}$) is subtracted.

(2) The redistribution of force to the neighbors is proportional to the force in the relaxing site. In the Feder and the BTW models this is not the case: A constant amount (usually 1) is transferred to the nearby sites.

(3) The BTW model is conservative, unlike the Feder model, where the amount of nonconservation is $F_{ij} - F_{th}$, where $F_{ij}$ is the relaxing site. The nonconservation in our model is proportional to $F_{ij}$.

(4) This model, as well as the Feder model and the model suggested by Bak and Tang [12] and Bak and Chen [13] for description of earthquakes, is globally driven, i.e., the full bulk is raised simultaneously. This driving is a natural choice for models representing earthquakes or other globally driven systems. This is a very important point since random local driving can destroy the correlations and the self-organization of the system. Furthermore, we must note that we see no direct relationship between the latter three models and the two-dimensional Burridge-Knopoff spring-block model.

(5) If $K_1 = K_2$ ($\alpha_1 = \alpha_2$), this model is also anisotropic. We can control the amount of anisotropy by changing the ratio between $\alpha_1$ and $\alpha_2$. In the extreme case the system will be one dimensional.

### B. Scaling and SOC

It is very important to investigate how the finite size of the system affects the properties we measure. The trademark of SOC is the existence of a power-law distribution function of the avalanche sizes that scale with the system size. We concentrate our effort on the avalanche-size distribution, where the size is defined by the total number of relaxations in a single avalanche. It was found that this number is also proportional to the released strain (energy) in the system. Let $P(E,L)$ be the probability density of having an avalanche of size $E$ in a system of linear size $L$. If the distribution function (note that we use the concept "distribution function" as equivalent to "probability density" although the two terms are not mathematically equivalent) is a power law, we define the power-law exponent $B$:

$$P(E,L) \sim E^{-(1+B)}.$$  

The scaling properties of the system are investigated by finite-size-scaling analysis—that is, we make the ansatz that the probability density scales with system size as

$$P(E,L) = L^{-\beta g(E/L^\nu)},$$

where $g$ is a so-called universal scaling function and $\beta$ and $\nu$ are critical indices describing the scaling of the distribution function. The critical index $\nu$ expresses how the finite-size cutoff scales with system size, while the critical index $\beta$ is related to the normalization (or rather renormalization) of the distribution function.
FIG. 2. The phase diagram for the model. When the level of conservation \(2\alpha_1 + 2\alpha_2 < 1\), the system is nonconservative. If \(\alpha_1 = \alpha_2 = \alpha\), the model is isotropic. Finally, when \(\alpha_1 = 0, \alpha_2 \neq 0\), the system is purely one-dimensional.

If the distribution function is a power law, it is easy to show the following relation between the exponents (see, for example, Ref. [6]):

\[
1 + B = \frac{\beta}{\nu}
\]

We will measure the three exponents \(B, \nu, \) and \(\beta\) systematically in different points \((\alpha_1, \alpha_2)\) of the phase diagram of the model (see Fig. 2).

C. Definition of boundary conditions

The boundary is an integrated part of any finite system. The model would not be well defined without a specification of the boundary conditions. There is a very large range of possible boundary conditions. We can control the properties of the boundary by changing the \(\alpha\) at the boundary, henceforth called \(\alpha_{BC}\). As we will show later, the system is sensitive to the nature of the boundary.

For the spring-block problem there are two extreme possibilities: (a) The blocks in the boundary layer are connected only to blocks within the faults, implying \(\alpha_{BC} = \alpha / (1 - \alpha)\); the boundary is free. (b) The blocks in the boundary layer are coupled to an imaginary boundary block by springs, \(\alpha_{BC} = \alpha\); the boundary is open. We refer to Appendix B for a detailed discussion of those boundary conditions. Notice that the free boundary conditions are more conservative than the open boundary conditions. Neither the free nor the open boundary conditions are probably physically realistic, but the actual boundary conditions must be in between the two extreme limits. In the context of continuous cellular-automata models, one can even introduce a totally conservative boundary condition, \(\alpha_{BC} = \frac{1}{2}\), which we will denote as a reflecting boundary.

III. SIMULATION OF THE MODEL

The rules for the driving of our model are motivated by the dynamics of earthquakes. There are two time scales involved. One is defined by the motion of the tectonic plates, and the other is the actual duration of an earthquake. Since the first time scale is much larger than the second, we can separate the time scales. We consider the earthquake as instantaneous and do not drive the system during an avalanche. Thus the algorithm for simulating the system is the following: Define random initial forces in the system. Strain is accumulated uniformly across the system as the rigid plates move. When the force on a certain site is above the threshold value \(F_{th}\), the site will relax according to Eqs. (2) and (3). The triggered earthquake will stop when there are no sites left with a force above the threshold. Strain starts to accumulate once again. The system organizes into the critical state after a transient time which is proportional to the system volume but dependent on the level of conservation. We collect the statistics only after the system is organized. We continue this process to get proper statistics of the distribution function of the energy released (which is proportional to the total number of relaxations) during the earthquakes. The results that are presented in this

FIG. 3. (a) Simulation results for a 35×35 system with open boundary conditions. Different curves refer to different levels of conservation. The slopes of the curves become steeper as the \(\alpha\) values are decreased. The graphs correspond to \(\alpha = 0.25, 0.20, 0.15, 0.10, 0.075,\) and \(0.025\), respectively. Notice that the last point is not critical. (b) The power-law exponent \(B\) as a function of the elastic parameter \(\alpha\) defined in Eq. (3). Below \(\alpha = 0.05\), there is a transition to exponential decay. The arrows indicate the actual measured \(B\) values for earthquakes [15].
paper are based on statistics derived from 10 000 000–50 000 000 avalanches.

A. The isotropic model with open boundary condition

We discuss in this section the behavior of the system with open boundary conditions along the line $\alpha_1 = \alpha_2 = \alpha$ (see Fig. 2). The model displays SOC for $\alpha \geq 0.05$. However, the critical exponents change as a function of $\alpha$. We present the dependence of the exponent $\nu$ on $\alpha$ in Fig. 3. We did a detailed finite-size scaling for the values $\alpha \geq 0.10$. Below this $\alpha$ value, the exponent $\nu$ is too large to obtain proper statistics of the cutoff behavior. We present in Fig. 4 four examples of finite-size scaling for the $\alpha$ values 0.25, 0.245, 0.20, and 0.15, respectively. As seen in those graphs, finite-size scaling works very well. This verifies the criticality of the system. No correlation length is introduced by the nonconservation. We show in Fig. 5 results for the dependence of the critical exponents $\nu$ and $\beta$ on $\alpha$ in the region 0.10 $\leq \alpha \leq 0.25$.

Since the avalanches are completely localized for $\alpha = 0$, we know that a localization transition should occur at some $\alpha \geq 0$. Indeed, we see such a transition at approximately $\alpha = 0.05$. Below this level, the distribution function becomes localized. A system-size-independent exponential length scale appears in the distribution of avalanche sizes. The transition seems discontinuous, i.e., the length scale does not change continuously near the transition.

The exponent $\beta$ changes continuously in the range $0.05 \leq \alpha \leq 0.25$. However, for the scaling indices we see a very sharp transition between the conservative and nonconservative regimes. The scaling index $\nu$ drops from 3.3 to 1.8 when changing $\alpha$ from 0.25 to 0.245. We attribute this change of scaling to a transition in the temporal behavior of the avalanches. Introduce the integrated amount of local activity in the avalanche as a third dimension. When the system is conservative, the avalanches have the shape of a cone, where the height is approximately of the same scale as the radius. However,
TABLE I. The critical exponents for the isotropic system with open boundary conditions. The scaling relation, Eq. (6), is fulfilled within numerical accuracy (except for $\alpha=0.22$). For comparison we also list the critical exponents for the original BTW model.

<table>
<thead>
<tr>
<th>Model</th>
<th>$1 + B$</th>
<th>$\nu$ $\pm0.10$</th>
<th>$\beta$ $\pm0.10$</th>
<th>$\beta/\nu$ $\pm0.10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha=0.25$</td>
<td>1.22±0.05</td>
<td>3.3</td>
<td>4.2</td>
<td>1.27±0.05</td>
</tr>
<tr>
<td>$\alpha=0.2499$</td>
<td>1.22±0.05</td>
<td>3.1</td>
<td>3.95</td>
<td>1.27±0.06</td>
</tr>
<tr>
<td>$\alpha=0.245$</td>
<td>1.17±0.05</td>
<td>1.7</td>
<td>2.2</td>
<td>1.29±0.11</td>
</tr>
<tr>
<td>$\alpha=0.24$</td>
<td>1.25±0.05</td>
<td>1.8</td>
<td>2.6</td>
<td>1.44±0.11</td>
</tr>
<tr>
<td>$\alpha=0.22$</td>
<td>1.65±0.05</td>
<td>2.0</td>
<td>3.7</td>
<td>1.85±0.13</td>
</tr>
<tr>
<td>$\alpha=0.20$</td>
<td>1.89±0.10</td>
<td>2.2</td>
<td>4.25</td>
<td>1.93±0.12</td>
</tr>
<tr>
<td>$\alpha=0.175$</td>
<td>2.06±0.10</td>
<td>2.35</td>
<td>4.8</td>
<td>2.04±0.12</td>
</tr>
<tr>
<td>$\alpha=0.15$</td>
<td>2.22±0.10</td>
<td>2.35</td>
<td>5.15</td>
<td>2.19±0.13</td>
</tr>
<tr>
<td>$\alpha=0.10$</td>
<td>2.72±0.10</td>
<td>2.3</td>
<td>6.6</td>
<td>2.87±0.18</td>
</tr>
<tr>
<td>BTW, locally</td>
<td>1.1±0.05</td>
<td>2.0</td>
<td>2.3</td>
<td>1.15±0.08</td>
</tr>
<tr>
<td>BTW, globally</td>
<td>1.1±0.05</td>
<td>2.0</td>
<td>2.3</td>
<td>1.15±0.08</td>
</tr>
</tbody>
</table>

when the system is nonconservative, even the smallest dissipation level would define a maximal number of relaxation at a given site. Therefore, the avalanches will become flat. This implies a change in the scaling behavior of the avalanches from $3$ to $2$. We believe that the width of the transition will depend on the system size. Another interesting feature in the nonconservative region of the model is that although the exponent $B$ changes strongly, $\nu$ is almost a constant. Still, the relationship between the exponents, Eq. (6), is fulfilled in all the measured region.

It is worth noting that at each relaxation in the Feder model (with dynamical noise and $F_{th}=4$), the effective $\alpha$ is $1/F_{\nu_j}$. Thus we can estimate the level of conservation by measuring the average over a large number of relaxations. We find $\alpha\approx0.217$, and indeed, the exponent and the scaling indices are consistent with the point $\alpha\approx0.217$ in our model (with open boundary conditions). Another interesting comparison is with the BTW model with global and local driving. We present all the results in Table I. While the difference in $B$ is very small, there is a dramatic change in the scaling exponents $\nu$ and $\beta$.

B. Different boundary conditions

The data that we presented in Sec. IIIA imply that there is a dependence of the exponents on the parameters of the model. It is also well known that existence of criticality depends on the boundary conditions. A model with BTW dynamics and reflecting boundary conditions (conservative, for example), cannot display any critical behavior. However, no systematic study of this issue was done even for the original BTW model. In this section we show that the results depend very strongly on the boundary conditions.

First, we choose the physically interesting case of the free boundary conditions. In Fig. 6 we show the change of the exponent $B$ as the level of conservation is changed for the model with free boundary conditions (note that the free boundary conditions are totally reflective for the conservative case so the model obviously cannot display criticality). In Fig. 7 we display the dependence for both the free and the open boundary conditions. The change in the boundary conditions induces a dramatic change in

![FIG. 6. Simulation results for a 35×35 system with free boundary conditions. Different curves refer to different levels of conservation. The slopes of the curves become steeper as the $\alpha$ values are decreased. The graphs correspond to $\alpha=0.245, 0.20, 0.15, 0.10, 0.05,$ and $0.01$, respectively.](image)

![FIG. 7. The power-law exponent $B$ as a function of the elastic parameter $\alpha$ defined in Eq. (3). Solid symbols correspond to the model with open boundary conditions. The measured $B$ values for the model with free boundary conditions are displayed as open symbols.](image)
the exponents. The exponents for the free boundary conditions are much lower than for the open boundary conditions. Also, the localization transition for the free boundary conditions is lowered to $\alpha \approx 0.01$.

We did a finite-size-scaling analysis for one point along this line, for $\alpha = 0.20$. Finite-size scaling seems to work very well in this case also, as is shown in Fig. 8.

To check this dependence more carefully, we changed the boundary conditions at the edge continuously from open boundary conditions $\alpha_{BC} = \alpha$ to reflecting boundary conditions $\alpha_{BC} = \frac{1}{2}$. We see a continuous change of the exponent $B$ when going from the open boundary conditions, $\alpha_{BC} = 0.20$ to a value of $\alpha_{BC} = 0.28$, which is larger than the $\alpha_{BC} = \alpha(1-\alpha) = 0.25$ used for free boundary conditions. Above this value, the systems seems to be noncritical in the sense that no power laws are seen. The distribution function can no longer be scaled with system size, although the size of the maximal avalanches scale with system size. We have observed similar behavior for other bulk values of $\alpha$, but we did not make a detailed study.

IV. RELATION TO EARTHQUAKES

In nature, earthquakes are probably the most relevant paradigm of self-organized criticality. In 1956 Gutenberg and Richter realized that the rate of occurrence of earthquakes of magnitude $M$ greater than $m$ is given by the relation

$$\log_{10}N(M > m) = a - bm,$$

where $a$ and $b$ are constants for a given fault. This is the Gutenberg-Richter law [14]. Measurements of the parameter $b$ yield a wide range of values for different faults. Values of $b$ from 0.80 to 1.06 for small earthquakes and 1.23 to 1.54 for large earthquakes have been recorded [15].

The energy (seismic moment) $E$ released during the earthquake is believed to increase exponentially with the earthquake magnitude,

$$\log_{10}E = c + dm,$$

where the parameter $d$ is 1 and $\frac{1}{4}$ for small and large earthquakes, respectively [16]. Thus the Gutenberg-Richter law is transformed into a power law for the number of observed earthquakes with energy greater than $E$:

$$N(E_0 > E) \approx E^{-b/d} = E^{-B},$$

Note that $B$ is in the same range for both small and large earthquakes, namely, 0.80−1.05, but it is not a universal exponent. In our model we measured the distribution function, Eq. (4). This explains why we use the notation $(1+B)$ for the power-law exponent.

The idea of explaining the Gutenberg-Richter law by such a two-dimensional model was already proposed by Otsuka [17]. It was also suggested independently by several authors [12,13,18–20] immediately after the introduction of the idea of SOC by Bak et al. that SOC might be a good explanation for the observed power laws. Most of those suggested models are conservative and have no physical interpretation in the context of the driven spring-block model. Furthermore, since the models are conservative, they predict unique power-law exponents which are much lower than the observed values ($B \approx 0.10$).

If the different elastic constants $K_1$, $K_2$, and $K_L$ are within the same scale, the characteristic value of $\alpha$ for earthquakes is 0.20, which amounts to a $B$ value around 0.9 in the case of open boundary conditions. The reported values of $B$ are indeed in this range. Thus our results, apart from providing an explanation for the observed power laws, also give some explanation for the observed variability. One should not look for universal values of $B$ in nature.

Another interesting connection of this model to earthquakes is that, like real earthquakes, it displays spatiotemporal correlations between the earthquakes. Those aspects are discussed in some detail elsewhere [10]. This gives us good reason to believe that this simplistic picture has a real connection to the actual fault dynamics that leads to earthquakes.

V. THE ANISOTROPIC CASE

As mentioned earlier, we can control the anisotropy in the system while keeping the level of conservation constant. We changed the ratio $\alpha_x/\alpha_y$ from 1 to 0 while keeping $2\alpha_x + 2\alpha_y = C$, where $C$ is a constant (see Fig. 2). We measured the distribution function for fixed conservation levels from $C = 0.90$ to 0.50. The change in $B$ is negligible. The system is critical, except for the case of the one-dimensional system where the distribution function is no longer a power law. In Fig. 9 we show a representative example of distribution functions that are derived for $C = 0.80$. For this particular value of $C$ we also measured the scaling exponents. We present the data in Table II. The finite-size-scaling exponent $v$ changes continuously from 2.2 to 1. This is the signature of the rising anisotropy in the model. The surprising minimal change in the exponent $B$ is due to constant con-
FIG. 9. Size distribution functions for the anisotropic case scanned along a constant conservation level, $C = 0.80$ (see Fig. 2). The anisotropic ratios are $1, \frac{2}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{8},$ and $0$. They are easily recognized by the decrease of the cutoff in the maximal avalanche size. Notice that the one-dimensional distribution has no resemblance to the rest, even though the largest avalanches scale as $E$.

FIG. 10. Size distribution functions for the anisotropic case in a conservative model, $C = 1$. The anisotropic ratios are $1, \frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{1}{8},$ and $0$. Notice that the cutoffs are not modified as in the nonconservative model. The scaling is modified to 2 in the one-dimensional case.

VI. CONCLUSIONS

The main results in this paper are the following.

1) Nonconservative models can organize themselves to a critical state. Apart from a relatively small localized region, the nonconservative model displays SOC for all conservation levels. This is seen only for models that are driven globally. Local drive together with nonconservation introduces a system-size-independent length scale (i.e., if the local drive is so large that it destroys the correlation in the lattice).

2) The self-organization is very sensitive to the boundary conditions. The critical exponents and scaling indices change as the boundary conditions are modified. However, this model displays SOC for a wide range of possible boundary conditions.

3) While SOC is a very stable feature of this model, the exponents are not universal. The exponent $B$ changes continuously when changing the level of conservation. Also, the boundary has an effect on the power-law exponent $B$. However, the anisotropy does not influence the value of $B$ as long as the level of conservation is constant. The scaling exponents are changed in all the above cases.

4) We observe several phase transitions in the system. We see a transition when going from a conservative to a nonconservative system. It manifests itself by a discontinuous change in the critical indices $\nu$ and $\beta$. The observed width of the transition is a finite-size effect. Another transition takes place at low $\alpha$ when the avalanches become localized. Finally, a transition from a critical to a noncritical system is seen when the anisotropy grows very large (the system becomes effectively one dimensional).

Clearly, the existence of criticality in nonconservative models implies that strong correlations are induced in the system by a self-organization process. Otherwise, avalanches would become completely localized for any nonconservation. To illustrate the self-organization procedure, we present the running average of earthquake

---

TABLE II. The critical indices for the anisotropic system with open boundary conditions. Note that even though Fig. 9 shows an almost negligible change in the power-law exponent, the ratio $\beta/\nu$ varies a lot and does not fulfill the scaling relation, Eq. (6).

<table>
<thead>
<tr>
<th>Model</th>
<th>$\nu$ ($\pm 0.10$)</th>
<th>$\beta$ ($\pm 0.10$)</th>
<th>$\beta/\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>($\alpha_1, \alpha_2$) = (0.20, 0.20)</td>
<td>2.2</td>
<td>4.25</td>
<td>1.93$\pm$0.12</td>
</tr>
<tr>
<td>($\alpha_1, \alpha_2$) = (0.15, 0.25)</td>
<td>2.0</td>
<td>4.4</td>
<td>2.37$\pm$0.13</td>
</tr>
<tr>
<td>($\alpha_1, \alpha_2$) = (0.10, 0.30)</td>
<td>1.6</td>
<td>3.8</td>
<td>2.37$\pm$0.21</td>
</tr>
<tr>
<td>($\alpha_1, \alpha_2$) = (0.05, 0.35)</td>
<td>1.1</td>
<td>2.8</td>
<td>2.55$\pm$0.33</td>
</tr>
</tbody>
</table>
sizes in Fig. 11. We start from a random uncorrelated configuration. The average grows gradually, indicating the autocorrelation of the system into the critical state. Since the system autocorrelates slowly, the maximal sizes of avalanches grow slowly. This growth is limited only by the system size, which defines the avalanche-size cutoff. On the other hand, if we start with a correlated system, the average will immediately stabilize into the system-size-dependent value. This is also shown in Fig. 11.

The avalanches are generated by correlated clusters of sites, which in turn are generated by the avalanches. The correlated clusters are modified by interactions with other clusters through avalanches. Another organization process is that the boundary acts as a source of correlation. The evidence for this is the effect of boundary conditions on the critical exponents.

A change in the parameters of the models will not destroy the self-organization process, but will modify it strongly. That is the origin for our results. A more detailed picture of this organization can be easily derived in one dimension, where the boundary between clusters is a point and not a line.

With this in mind, it is easy to understand why no criticality is seen for models, which are driven locally. The local perturbations will destroy the correlations in the system. This interpretation can be tested numerically in our model. The system should be resistant to noise, which will not destroy the correlations. On the other hand, the model should display localization if the noise is so large that it can destroy the correlations. And indeed it does behave in this way.

We have proved the nonuniversality of the self-organization process for our model. However, as we noted before, exponents will be modified even for the original BTW model when the boundary condition is changed or when introducing symmetry-preserving modifications in the algorithm. So our basic conclusions seem to also be relevant to other self-organizing systems. We believe that the main theoretical effort in this field should be dedicated to the understanding of the common features of the self-organization, rather than to looking for universality in the results.

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APPENDIX A

Assume the force on a block at position \((i,j)\) is above the threshold value, that is
\[
F_{i,j} \geq F_{i,j} = K_1 [2x_{i,j} - x_{i+1,j} - x_{i+1,j+1}] \\
+ K_2 [2x_{i,j} - x_{i,j-1} - x_{i,j+1}] \\
+ K_L x_{i,j} ,
\]
(A1)

If \(x_{i,j}\) denotes the displacement of block \((i,j)\) from the relaxed position after the block has slipped to zero-force position, then
\[
0 = K_1 [2x_{i,j} - x_{i-1,j} - x_{i+1,j}] \\
+ K_2 [2x_{i,j} - x_{i,j-1} - x_{i,j+1}] + K_L x_{i,j} ,
\]
(A2)

where we exploit the fact that nearest-neighbor blocks of \((i,j)\) cannot be supercritical at the same time, i.e.,
\[
x_{i\pm 1,j} = x_{i\pm 1,j} ,
\]
(A3)

\[
x_{i,j\pm 1} = x_{i,j\pm 1} .
\]

The slipping block \((i,j)\) affects the force on nearest-neighboring blocks. As an example, we calculate the change of force on block \((i,j+1)\). The force on block \((i,j+1)\) is
\[
F_{i,j+1} = K_1 [2x_{i,j+1} - x_{i-1,j+1} - x_{i+1,j+1}] \\
+ K_2 [2x_{i,j+1} - x_{i,j} - x_{i,j+2}] \\
+ K_L x_{i,j+1} .
\]
(A4)

Thus the change of force due to a slip at position \((i,j)\) is
\[
\delta F_{i,j+1} = - K_2 d x_{i,j} .
\]
(A5)

Notice that the force on block \((i,j+1)\) may very well be affected by a slip at position \((i,j+2)\) but that does not interfere with this argument. An expression for the change
in displacement of block \((i,j)\),
\[
dx_{i,j} = x_{i,j} - x_{i,j}.
\]
\[\text{(A6)}\]
is obtained by subtracting Eq. (A1) from Eq. (A2),
\[
0 - F_{i,j} = [2K_1 + 2K_2 + K_L] dx_{i,j}.
\]
\[\text{(A7)}\]
Finally, substituting Eq. (A7) into Eq. (A5), we find
\[
\delta F_{i+1,j} = \frac{K_2}{2K_1 + 2K_2 + K_L} F_{i,j}.
\]
\[\text{(A8)}\]

\section*{Appendix B}

The boundary condition is free if the blocks in the boundary layer are connected only to blocks within the fault, i.e., the force on a boundary block, say, at site \((i, L)\), is given by
\[
F_{i,L} = K_1 [2x_{i,j} - x_{i-1,j} - x_{i+1,j}]
+ K_2 [x_{i,j} - x_{i,j-1}] + K_L x_{i,j}.
\]
\[\text{(B1)}\]
If this blocks slips, we find
\[
0 - F_{i,L} = [2K_1 + K_2 + K_L] dx_{i,L},
\]
\[\text{(B2)}\]
resulting in
\[
\delta F_{i\pm 1,L} = \frac{K_1}{2K_1 + K_2 + K_L} F_{i,L} = \alpha_{BC} F_{i,L}.
\]
\[\text{(B3)}\]
If the model is isotropic, \(K_1 = K_2 = K\), then
\[
\alpha = \frac{K}{4K + K_L}.
\]
\[\text{(B4)}\]
We can express the elastic ratio \(\alpha_{BC}\) used when boundary blocks slip in terms of the bulk \(\alpha\),
\[
\alpha_{BC} = \frac{\alpha}{1 - \alpha}.
\]
\[\text{(B5)}\]
If a block in one of the corners slips, we use
\[
\alpha_{BC} = \frac{\alpha}{1 - 2\alpha}.
\]
\[\text{(B6)}\]

For a simulation of the model with open boundary conditions, we use the same \(\alpha\) all over the lattice. That is, if block \((i, L)\) slips, we increase the force on the three neighboring blocks \((i\neq 1, L)\) with an amount equal to \(\alpha_{BC} F_{i,L}\), where \(\alpha_{BC} = \alpha\) is defined in Eq. (3).

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Appendix D


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*Deterministic* $1/f$ noise

*in*

*nonconservative models.*
Deterministic 1/f Noise in Nonconservative Models of Self-Organized Criticality

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(Received 24 January 1992)

Generic, deterministic, nonconservative models displaying self-organized criticality are shown to exhibit 1/f noise. The exponent of the power spectrum depends on the level of conservation.

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One of the great mysteries of physics is the 1/f noise observed in signals from sources ranging from the light of quasars to the flow of the river Nile and the current flowing through a resistor [1,2]. The noise is called “1/f” noise despite the fact that the exponent of the power spectrum is rarely 1, but varies from system to system, and is typically in the range of 0.6 to 1.6. Remarkably, the power law is obeyed over several decades, spanning time scales where one might expect the physics to differ significantly. Using a very simple dynamical model to capture the essence of the underlying mechanism, we argue and demonstrate numerically that 1/f noise is a deterministic self-organized critical phenomenon emerging naturally in interactive dissipative dynamical systems with many degrees of freedom.

A few years ago, Bak, Tang, and Wiesenfeld (BTW) [3] discovered that extended dynamical many-body systems self-organize into a critical state. The idea complements the concept of “chaos” wherein simple systems, with a small number of degrees of freedom, can display quite complex behavior. The critical state is characterized by avalanches (activity) with power-law spatial and temporal distribution functions limited only by the size of the system. The original models were driven by white noise, but later it was shown that deterministic models exhibit the same behavior, that is, the criticality is not caused by but, on the contrary, is robust with respect to noise. Thus, a mechanism for establishing coherence over all time scales was provided.

However, the spatiotemporal scaling in the self-organized critical state does not necessarily manifest itself in nontrivial exponents of the power spectrum. Jensen, Christensen, and Fogedby [4] and later Kertész and Kiss [5] showed that the power spectrum of the activity was in fact 1/f², i.e., the spectrum of a random walk. The proper relationship between the joint probability density \( P(S=s, T=t) \) of having an avalanche of size \( s \) and lifetime \( t \) and the power spectrum \( S(f) \) of linearly superimposed avalanches was worked out by Jensen, Christensen, and Fogedby [4(a)] and later generalized by them [4(b)]. Introducing the weighted lifetime distribution

\[
\Lambda(t) = \sum_s s^2 P(S=s, T=t),
\]

and assuming that \( \Lambda(t) \) exhibits a scaling behavior

\[
\Lambda(t) \sim t^{\alpha}, \quad 0 < t_1 \leq t \leq t_2 < \infty,
\]

and is negligible outside this interval, they showed

\[
S(f) \sim \begin{cases} 
1, & \mu + 1 < 0, \\
\frac{1}{(\mu+1)!} f^{-(\mu+1)}, & 0 < \mu + 1 < -\alpha_\infty, \\
\frac{2}{\alpha_\infty}, & \mu + 1 > -\alpha_\infty,
\end{cases}
\]

for the scaling of the power spectrum in the intermediate frequency regime \( 1/2\pi t_2 \ll f \ll 1/2\pi t_1 \). Here, \( \alpha_\infty \) is an exponent characterizing the shape of the avalanches and it can be proven to be \( \leq -2 \). For \( f \to 0 \) the power spectrum becomes white, since a linearly superimposed signal cannot contain temporal correlations beyond the longest possible lifetime of an avalanche. Thus, in order to get nontrivial exponents, \( \mu + 1 \) must be between 0 and 2, whereas in the BTW model \( \mu = 3.1, 2.77, 2.69 \), and 2.56 in dimension 2, 3, 4, and 5, respectively. Notice that the lower frequency cutoff scales inverse proportionally with the upper temporal cutoff \( t_2 \) of the weighted lifetime distribution.

In the BTW model the local dynamical variable was conserved during the relaxations, whereas realistic models of natural phenomena, such as earthquakes, do not typically have any inherent conservation law. For some time it was believed that the introduction of any degree of nonconservation into a BTW-type model would necessarily lead to a finite correlation length [6]. Recently, however, Feder and Feder [7] introduced a two-dimensional, nonconservative model displaying criticality. Shortly afterwards Olami, Feder, and Christensen [8] discovered a class of deterministic models, related to spring-block models of earthquakes, which exhibit self-organized critical behavior with a conservation level down to 20%, and nonuniversal exponents depending on the level of conservation.

The simplicity and robustness of those models suggest that they can be viewed as generic “Ising” models of dissipative many-body systems; hence, we concentrate our effort on those models: A set of dynamical variables \( F_{i,j} \), representing the local force (strain) at site \((i,j)\), is defined on a two-dimensional lattice \( 1 \leq i, j \leq L \). The values of \( F_{i,j} \) are increased uniformly at an essentially infinitely slow rate until somewhere the force exceeds a critical value \( F_\text{th} \). Then the force on the unstable site, \( F_{i,j} \), and the values of the force at its nearest neighbors \( F_{nn} \), are updated according to the simple relaxation rule

\[
F_{nn} \rightarrow F_{nn} + \alpha F_{i,j}, \quad F_{i,j} \rightarrow 0.
\]

This initiates an avalanche which lasts for \( t \) time units (a unit time step is defined as one simultaneous update of

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the whole lattice) and involves $s$ relaxations. As $F$ keeps increasing, there will be more avalanches triggered by instability on other sites. In the beginning, correlations are short range in time and space. During a long transient period, depending on the size of the system and $\alpha$, the correlations grow, until saturating at a value limited by and scaling with the size of the system, indicative of the slow self-organizing process leading to the stationary critical state. Notice that the level of conservation is given by $4\alpha$.

The difference from the deterministic, continuous version of the original BTW model may seem subtle, but nevertheless has striking consequences. In the BTW model the decrease in the value of the relaxing site was a fixed amount (say 4) and the transfer to the neighbors was independent of the previous state of the system. This rigid way of transferring force did not allow the system to self-organize except for the special case of a conservative system. In the model given above the relaxing amount and the transfer to the neighbors depend on the state of the system. This appears to be a much more general situation. The fact that one has a handle on the exponents, through the level of conservation, suggests that a power spectrum with nonuniversal nontrivial exponents may emerge, as observed in nature.

We now focus on the weighted lifetime distribution for the avalanches defined in Eq. (1) and its relation to the power spectrum of linearly superimposed signals. This is to mimic the total activity in a very large, slowly driven system in which we can neglect the interference between different avalanches.

Figure 1 shows the weighted lifetime distribution $\Lambda(t)$ for various values of $\alpha$. Indeed, they obey power laws with the exponent $\mu$ depending on the dissipation. The values of $\mu$ are in the range where nontrivial exponents for the power spectra are expected.

In order to measure directly the power spectrum of linearly superimposed avalanches we generate realizations of the number of relaxations per unit time step $f(\tau)$ by repeating the following procedure: (1) Set $f(\tau) = 0$ for all $\tau$. (2) Perturb the system until an avalanche is activated. (3) Choose a starting time $\tau_0$ at random, and increase $f(\tau_0 + \tau)$ by the activity (number of relaxations per unit time step), for $\tau = 1, \ldots, t$, where $t$ is the lifetime of the avalanche. (4) Go to (2).

The assumption of no interference between different

![Figure 1](image)

**FIG. 1.** Distribution of weighted duration of avalanches in a system of size $L = 100$. (a) $\alpha = 0.10$, (b) $\alpha = 0.15$, and (c) $\alpha = 0.20$. The measured exponents $\mu$ are $-0.40$, $0.61$, and $0.92$, respectively.

---

**Table 1.** The measured exponents $\mu + 1$ and $\varphi$ for the weighted lifetime distribution and power spectrum, respectively. The exponents are, within numerical accuracy, consistent with Eq. (3). For $0.0225 < \alpha \leq 0.25$ it was not possible to identify any scaling region. For comparison we also list the critical exponents for the original two-dimensional BTW model.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\mu + 1$</th>
<th>$\varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha = 0.10$</td>
<td>$0.60 \pm 0.10$</td>
<td>$0.50 \pm 0.10$</td>
</tr>
<tr>
<td>$\alpha = 0.125$</td>
<td>$1.20 \pm 0.10$</td>
<td>$1.18 \pm 0.05$</td>
</tr>
<tr>
<td>$\alpha = 0.15$</td>
<td>$1.61 \pm 0.10$</td>
<td>$1.56 \pm 0.05$</td>
</tr>
<tr>
<td>$\alpha = 0.175$</td>
<td>$1.80 \pm 0.10$</td>
<td>$1.76 \pm 0.05$</td>
</tr>
<tr>
<td>$\alpha = 0.20$</td>
<td>$1.92 \pm 0.10$</td>
<td>$1.80 \pm 0.05$</td>
</tr>
<tr>
<td>$\alpha = 0.225$</td>
<td>$2.24 \pm 0.10$</td>
<td>$1.92 \pm 0.05$</td>
</tr>
<tr>
<td>2D BTW</td>
<td>$3.10 \pm 0.10$</td>
<td>$2.00 \pm 0.05$</td>
</tr>
</tbody>
</table>
FIG. 2. The power spectrum of randomly superimposed avalanches for the same \( \alpha \) values as in Fig. 1. The arrow indicates the upper frequency cutoff \( 1/2\pi \) for the scaling region. The exponents \( \varphi \) of the power spectrum are (a) \( \varphi = 0.50 \), (b) \( \varphi = 1.56 \), and (c) \( \varphi = 1.80 \). Within numerical accuracy, those values are consistent with the values of \( \mu \). (d) The power spectrum of a system of size \( L = 250 \) driven continuously at a rate \( p = 0.001 \) for \( \alpha = 0.20 \). The straight line has a slope of \(-1.93\).

avalanches was essential for the derivation of Eq. (3). Time sequences \( j(\tau) \) generated according to the algorithm above, will, of course, fulfill this requirement.

For a given realization \( j(\tau) \) we make the Fourier transform \( \hat{j}(\omega) \), and define the power spectrum

\[
S(\omega) = |\hat{j}(\omega)|^2,
\]

which is a strongly fluctuating function of the frequency \( f \). We average over many different realizations (up to 1000 realizations) to reduce the variance in the power spectrum, which for one single realization is a 100% standard deviation. Figures 2(a)–2(c) display the resulting power spectra measured by randomly superimposing the avalanches in a system of size \( L = 100 \). We recall that the frequency region under consideration is \( 1/2\pi < f < \infty \), since \( f_1 = 1 \) by definition. The slopes of the straight lines are indeed in agreement with the predicted values obtained from substituting the values from Fig. 1 into Eq. (3). Table I lists all our simulation results. The exponents are roughly in the regime observed experimentally. Note the change in the sign of the slope \( \mu \) is related to a change of the power-law exponent from values less than unity to values greater than unity.

Alternatively, the time sequence \( j(\tau) \) can be generated by a direct measurement of the activity in a slowly driven system of considerable size. Figure 2(d) presents the power spectrum of a system of size \( L = 250 \) driven with a finite rate \( p = 0.001 \) for \( \alpha = 0.20 \). The long-term correlations ignored by the random superposition method do not affect the shorter time scales of interest here. Indeed, the power spectrum seems to be in even better agreement with the exponent extracted from the generator \( \mu \) of the weighted lifetime distribution.

Finally, in order to verify that the interval in which the power spectrum displays \( 1/f \) noise scales with system size, we generated the power spectrum for various system...
FIG. 3. The power spectrum for systems with \( \alpha = 0.20 \) but different system sizes, \( L = 45, 70, 100 \). The lower frequency cutoff scales with system size, while the upper frequency cutoff is a constant. The exponent of the power spectrum seems to change slightly with system size.

sizes, \( L = 45, 70, 100 \), for \( \alpha = 0.20 \). The power spectrum is shown in Fig. 3. We observe that the lower frequency cutoff scales with system size \( L \). This is a unique fingerprint of a many-body phenomenon. Hence, with this characteristic in mind, we urge that experiments be performed on systems of varying size in order to check the assertion that the \( 1/f \) noise is a critical dynamical many-body effect.

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Appendix E


Z. Olami and K. Christensen

Temporal correlations, universality, and multifractality

in a

spring-block model of earthquakes.
Temporal correlations, universality, and multifractality in a spring-block model of earthquakes

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We characterize the complex temporal structure of the earthquake activity in a nonconservative spring-block model of earthquakes. The temporal sequence of the occurrence of earthquakes is observed to be multifractal. The clustering (temporal correlations) between earthquakes is characterized by a universal exponent. Those correlations disappear when the model becomes conservative.

PACS number(s): 05.40.+j, 05.45.+b

The internal organization of driven nonequilibrium systems has received a lot of attention recently. The sandpile cellular automaton suggested by Bak, Tang, and Wiesenfeld (BTW) is an example of such a system [1]. BTW showed that it is representative of a certain class of conservative systems which drive themselves into a statistically stationary state, characterized by power-law spatial and temporal distribution functions. Hence, the systems have no intrinsic length or time scales and is in this sense critical. This type of behavior was named self-organized criticality (SOC).

The seismic system is an example of a physical system displaying power-law behavior. The power-law distribution for the earthquake intensity is interpreted as a signature of SOC: The movement of the tectonic plates drives the system into a critical state [2].

We have recently proposed a continuous, nonconservative cellular automaton to describe the dynamics of a driven spring-block system modeling earthquakes. This nonconservative model displays robust SOC over most of its parameter range. The critical exponents are not universal, but depend on the degree of the nonconservativeness and on the boundary conditions as presented in detail elsewhere [3].

The temporal behavior of this model is interesting because of several reasons. First, it is known that earthquakes display fractal clustering and correlated behavior (see [4] and references therein). For shallow earthquakes, the amount of clustering after large earthquakes is described by a universal power law: the Omori law. We have studied the temporal behavior of this dynamical model and shown that it displays clustering for the larger earthquakes. Second, since the model displays nonuniversality for the critical parameters related to the earthquakes, such as the $b$ value of the Gutenberg-Richter law, the question is whether there exists any universality for other critical exponents associated with temporal clustering. Finally, another interesting issue is the difference between the conservative model and the nonconservative model. There is a discontinuous transition of the scaling exponents when going from a conservative into a nonconservative system. However, there seems to be more fundamental changes in the temporal behavior, which are related to the fact that SOC in a nonconservative model can occur only if there are correlations between sites in avalanche clusters. The relative stability of those clusters are responsible for the observed clustering.

In this paper we focus on the temporal set of earthquakes which are generated by the deterministic dynamics of the spring-block model. We calculate the fractal dimension of the temporal sequence of earthquakes and find that it displays a multifractal behavior. We calculate the earthquake correlation function and show that it displays a universal behavior for critical nonconservative systems. The observed exponent is that reported in the Omori law. The total clustering associated with an earthquake depends on the size of the earthquake, the system size, and the amount of conservation. We characterize the scaling of this function. A system-size-dependent cutoff time appears in the problem; above this time no correlation is seen. We interpret those results as related to memory effects which are essential if any criticality is to be seen in such a system.

We first describe our model briefly. It can be derived from a two-dimensional spring block model for earthquakes [3]. It is defined on a two-dimensional lattice by a set of dynamical variables $F_{i,j}$ that represent the local force on a site $(i,j)$, where $1 \leq i, j \leq L$. The local forces increase uniformly at a very slow rate until one site reaches a threshold value $F_{th}$. Then the forces on the unstable site and its nearest neighbors are updated according to the simple relaxation rule

$$F_{NN} \rightarrow F_{NN} + a F_{i,j},$$

$$F_{i,j} \rightarrow 0.$$  (1)

This initiates an avalanche which either stops immediately or propagates further in the system. The distribution function of the size of the avalanches is a power-law distribution, where the cutoff scales with system size $L$, see Ref. [3]. The parameter $a$ is related to the spring constants of the spring model. Notice that this system is nonconservative if $a < 0.25$.

We can characterize the occurrence of each avalanche by an occurrence time defined by the driving of the system (the increase of strain is proportional to the elapsed time).

We discuss the earthquake temporal sequence $\Delta = \{E_i\}$, where $i_0$ is the occurrence time of the earthquake and $E$ is its energy. Subsets $\Delta_{\geq E_0} = \{E_i \in \Delta | E \geq E_0\}$ of this temporal sequence are defined by assigning a minimal energy...
$E_0$ for the earthquake under consideration, see Fig. 1.

We can ascribe a fractal dimension $D(E_0)$ to each subset $\mathcal{S}_{E_0}$ (see [5] for a detailed definition). We measure the number of intervals $N(\delta, E_0)$ of size $\delta$ needed to cover the subset $\mathcal{S}_{E_0}$ and we find

$$N(\delta, E_0) \sim \delta^{-D(E_0)}.$$  \hspace{1cm} (2)

The fractal dimension $D(E_0)$ depends on the energy $E_0$. Figure 2(a) is the result of counting intervals for a system of size $L = 70$ for $\alpha = 0.20$ for three different energies $E_0 = 690, 1608,$ and $2297$. Notice that the interval determining the fractal dimension grows with the minimal energy $E_0$. When the size of the system is enlarged the intervals belonging to the scaled avalanches grow like $L^\gamma$, $\gamma = 0.45 - 0.64$.

Figure 2(b) displays the measured fractal dimension as a function of the minimal energy $E_0$. The fractal dimension for small earthquakes is 1, which is the characteristic of a random set. The results in Fig. 2 are given only for energies where the interval is big enough to distinguish the fractal exponent. A nontrivial fractal dimension appears for larger earthquakes, indicating clustering.

In order to characterize the clustering more precisely we define an earthquake clustering function in the following way:

$$g(t) = \langle n(t) \rangle_{t_0} - t \langle n \rangle,$$ \hspace{1cm} (3)

where $\langle n(t) \rangle_{t_0}$ is the number of avalanches in the interval $(t_0, t_0 + t)$ averaged over all $t_0$ in $\mathcal{S}$, and $\langle n \rangle$ is the average density of earthquakes. $g(t)$ is simply related to an integral of a coarse-grained correlation function. For a Poisson process $g(t)$ is identically zero. We define $g_{E_0}(t)$ as $g(t)$ calculated for the subset $\mathcal{S}_{E_0}$ i.e., we restrict ourselves to the earthquakes with an energy release larger than $E_0$. The excess number of earthquakes a time $t$ after the avalanche is $dg(t)/dt$. It is also proportional to the temporal point-point correlation function. For large earthquakes this behavior was reported to be a $1/t$ behavior.

To measure this correlation we generated large data sets. Since we expect to see different clustering for large and small earthquakes we measured the functions $g_{E_0}(t)$ for the full range of energy. We give an example for the results in Fig. 3. The basic form of $g_{E_0}(t)$ for the long-term behavior is

$$g_{E_0}(t) \sim \begin{cases} \ln(t), & t < t_{co}(L), \\ f(E_0, L), & t > t_{co}(L). \end{cases}$$ \hspace{1cm} (4)

The basic logarithmic dependence does not depend on $\alpha$ in the nonconservative critical models. This behavior seems to be universal. The exponent of the Omori law will be the same for all nonconservative models, namely, $-1.0$. The temporal correlations of earthquakes in this model display universality even though the exponents for the frequency-energy distribution display no universality. The cutoff time $t_{co}(L)$ for the logarithmic behavior does
not depend on the minimal energy of the set, but it depends on $a$ and the system size $L$. We present results for $f(E_0, L)$ for $a=0.20$ and $L=35$ in Fig. 4. Below a minimal energy too much fluctuations are seen to distinguish any correlation. Notice that the drop at larger energies is a result of the fact that they are near the cutoff. We study the scaling of $f(E_0, L)$ with the system size $L$. The finite-size scaling hypothesis

$$f(E_0, L) = L^{-\beta} g(E/L^\gamma)$$

(5)

seems to work well with $\gamma=2.2$, which is the scaling exponent reported in [31], and $\beta=-1.6$.

We further checked the dependence of the cutoff time $t_{co}$ on $a$. For $a=0.25$ the cutoff time is zero, as the model becomes nonconservative a finite cutoff time appears. The cutoff increases with decreasing $a$. For $a=0.20$ we checked the dependence of $t_{co}$ on system size $L$, and it grows algebraically with $L$: $t_{co}(L) \sim L^{0.7}$. Thus, for $a<0.25$ this cutoff time will diverge with system size. Notice that the exponents describing the growth with system size of the scaling regions of phenomena related to the earthquake temporal sequence are small.

The conservative case is characterized by a complete lack of temporal correlations. This is a result of the conservative nature of the model. The avalanches are very big and multiple relaxations can occur at sites in the system during an avalanche. Since the avalanches are conservative the interoccurrence time is much smaller. Both effects drive the correlation time to zero. However, for nonconservative models the situation is very different. Since the model is nonconservative no multiple relaxations will occur during an avalanche. Moreover, such models can become critical only if internal correlations are created inside the lattice. Correlated clusters are created by the avalanches. On the other hand, those clusters will be modified by avalanches but not completely destroyed. This is the reason for the increase in the cutoff time as well as for the "memory" effects in the avalanches.

One should notice that the clustering observed in this model is for time scales which are larger than the buildup time of an earthquake. Thus the clustering we observe is a long-term clustering. The exact relationship between this and the Omori law for aftershocks (which might be related to instabilities generated by a large earthquake) is somewhat unclear. Similar calculations on earthquake data were done by Kagan and Jackson [4]. However, it is very hard to deduce any exact exponents from their data.

We have shown that nonconservative models for earthquakes display a complex multifractal behavior. The nonconservation imply the existence of strong correlations between earthquakes. The exponents governing this clustering are universal for nonconservative critical systems. Furthermore, the total clustering is related to the earthquake energy through the function $f(E_0, L)$. The transition to a conservative model is associated with a zero correlation time.

Other temporal measurements like point-point correlation function and probability distribution for interoccurrence times do not display universality.
K.C. gratefully acknowledges the financial support of Carlsbergfondet. Both authors appreciate the support and hospitality of Brookhaven National Laboratory. This work was supported by the Division of Basic Energy Sciences, U.S. DOE, under Contract No. DE-AC02-76CH00016.

Appendix F

Dansk resumé

Der er to almindeligt udbredte fænomener i naturen, som stadig mangler en fysisk forklaring. Det ene vedrører de rumlige strukturer, vi kalder fraktaler, medens det andet angår de tidslige fysiske signaler, der udviser $1/f$ støj [1, 2].

Fraktaler er rumlige strukturer med selv-similære egenskaber, d.v.s. de forandrer sig ikke, selv om man ændrer den længdeskala, hvormed man betragter dem. De selv-similære egenskaber udmønter sig i diverse potenslove for de relevante fysiske observable, hvilket er matematikkens måde at udtrykke fraværet af en karakteristisk længdeskala. Fraktale strukturer forekommer overalt i naturen, det være sig lige fra mønstre i snefnug og blomkål til stjernefordelingen på himlen, ja selv galaksefordelingen i universet udviser sådanne strukturer. Turbulens er et andet fysisk fænomen, det med rette kan kaldes fraktalt. Bl.a. dissiperes (eng. dissipates) energien ikke uniformt i væsken men intermittent (eng. intermittently) via kaskader af alle størrelser.

Det seismiske system er endnu et eksempel på, at et system med mange frihedsgrader udviser fraktale egenskaber: Fault systemet er skala-invariant; måler man antallet af faults som en funktion af deres længde, viser det sig at være udtrykt i en potenslov. Ligeledes er hyppigheden af jordskælv som funktion af den frigivne energi en potenslov, som er relateret til Gutenberg-Richter loven [3], se Figur 1.1(a). Den geometriske fordeling af epicentre er fraktal. Endvidere fremtræder også forekomsten af jordskælv med fraktal opførsel. Så tidlig som i 1894 fandt F. Omori, at antallet af jordskælv efter et stort jordskælv aftager som $1/t$, hvor $t$ er den tid der er gået siden det store jordskælv [4], se Figur 1.1(b). Foruden
disse kortrækkende tidslige korrelationer findes der også langtrækkende
tidslige korrelationer mellem de store jordskælv. De store jordskælv
forekommer hverken periodisk eller tilfældigt i tiden men har en ten-
dens til at arrivere i tidslige klynger [5].

Det seismiske system har mange lighedspunkter med det turbulente
system [6]: (1) begge er dynamiske systemer med mange vekselvirkende
frihedsgrader, (2) energien tilføres uniformt i begge systemer, (3) en-
ergien dissiperes over alle længde skaler, (4) energien dissiperes inter-
mittent i tiden og (5) begge systemer udviser både rumlig og tidslig
fraktal opførsel.

Det andet fænomen er rent tidsligt og er knyttet til den spektrale
analyse af de tidslige signaler. Det viser sig ofte, at de betragtede fysiske
signaler ikke blot indeholder nogle få karakteristiske frekvenser, men at
alle frekvenser indrages, således at effekten i et signal er proportional
med $f^{-\varphi}$, $\varphi \approx 1$, deraf navnet “en-over-f støj”. Dette fænomen er ob-
serveret i bl.a. lyset fra quasarer, strømmen i Nilen, og strømmen, der
løber igennem en modstand [2, 9, 10]. Mærkværdigvis opretholdes denne
potenslov over adskillige dekader i tiden, hvor man eller kunne forvente,
at fysikken ville ænde sig signifikant.

Det kan ikke være tilfældigt, at begge disse fænomener så at sige
dukker op overalt. Det er mere sandsynligt, at der forefindes en generel
forklaring på, hvorfor rumlige og tidslige fraktaler går så godt i spænd.

P. Bak, C. Tang og K. Wiesenfeld (BTW) foreslog nylig, at fraktaler
of $1/f$ spektra er hhv. de rumlige og tidslige karakteristika til en kritisk
dynamisk attraktor (herefter kaldet den kritiske tilstand) for systemer
med mange frihedsgrader [11]. Den kritiske tilstand er analog til de kri-
tiske tilstande ved termodynamiske faseovergange for systemer i ligevægt,
idede rumlige og tidslige tæthedsfunktioner udviser potenslovsopførsel.
Pointen er blot, at det er umødevendigt at finindstille en system parameter
som f.eks. temperaturen. Systemerne organiserer sig selv; den kritiske
tilstand er iboende dynamikken, som en attraktor. Derfor beskrives den
kritiske tilstand som værende selv-organiserende, og det “nye” fysiske
fænomen “selv-organiseret kritikalitet” kan vel vise sig at være den søgte
forklaringen på hhv. fraktaler og $1/f$ støj.

Denne ide indtager en meget central rolle i den nye æra indenfor
fysikken, hvor der fokuseres på, hvordan “kompleksitet fremkommer via
simpelhed” i modsætning til tidligere tiders forsøg på “at reducere kom-
pleksitet til simpelhed” for at bruge nogle vendinger af P. W. Ander-
son [12]. Fraktale vækst fænomener såsom DLA (diffusive limited aggregation) er et eksempel på, hvordan simple lokale dynamiske regler kan føre til en mangfoldighed af fysiske fænomener, deriblandt skala-invarians [2, 13]. Det fascinerende samspil mellem de komplekse strukturer, der spontant dannes i dynamiske systemer og de simple lokale dynamiske regler for vekselvirkningen mellem de mange frihedsgrader er uhyre interessant [14].

En sandbunke er prototypen på et dynamisk system med mange vekselvirkende frihedsgrader, der selv-organiserer til en kritisk tilstand: Lad os drysse sand ned på et bord. Vi tager det ene efter det andet sandskorn, som på et tilfældigt valgt sted drysses ned på bordet. I begyndelsen sker der ikke så meget; de neddryssede sandskorn ligger så at sige, hvor de blev lagt, men efter et stykke tid begynder vi at se konturerne af små lokale sandbunker, og vi bemærker undertiden små laviner, efter vi har tilføjet et nyt sandskorn. Disse små laviner har til formål at sænke den lokale hældning, som kun kan opnå en vis kvantitet.

Hvis vi holde op med at drysse sand falder systemet næsten ojeblikkeligt til ro i en metastabil tilstand. Fortsætter vi drysproceduren, vil sandbunkens hældning vokse, de små lokale sandbunker fusionerer til en stor sandbunde, og vi vil undertiden se stadig større og større laviner, som et respons på vores perturbationer. Efter et stykke tid vil sandbunken ikke gro mere. Hældningen af sandbunkes sider kan ikke overgå en kritisk hældning, lige meget hvor langt vi drysproceduren forlænger, det pådrysset sand vil udløse en lavine, som fører til, at sand transporteres ned ad siden og, hvis lavinen er stor nok, ud over kanten af bordet.


Variationen i den lokale hældning resulterer netop i, at det er umuligt at forudsige, hvad der sker, når vi tilfører et nyt sandskorn. Nogle gange forårsager det, at en stor lavine udløses, andre gange giver det ekstra sandskorn kun anledning til, at et par korn triller en smule ned og at-


Simulationer af denne simple model viste at systemerne virkelig driver sig selv ind i en stationær tilstand, der er karakteriseret ved, at fordelingen af lavine størrelsen samt deres varighed er beskrevet som potenslove. Derfor kaldes denne klasse af systemer for self-organiserede kritiske systemer.

Kompleksiteten i disse modeller er, som sagt, ikke et resultat af komplekse lokale dynamiske regler; kompleksiteten opstår som en syntese af den kontinuerlige lokale vekselvirkning mellem alle frihedsgraderne i systemet. Disse model-eksempler på dynamiske systemer, som genererer fraktale strukturer, kan være en mulig forklaring på, hvorfor fraktale strukturer synes alligevel at i de pages med få frihedsgrader.

Studiet af selv-organiserede kritiske systemer har i stor udstrækning været baseret på studiet af diverse cellulære automater. Langt største delen af disse repræsenterede bevarende systemer, hvor de simple dynamiske regler bevarer den dynamiske variabel. Der blev fremsat en hypotese, der udsiger, at bevarelse loven er nødvendig (og tilstrækkelig) til, at et dynamisk system selv-organiserer til en kritisk tilstand [15, 16]. Problemet vedrørende muligheden af ikke-bevarende systemer, der udviser selv-organiseret kritikalitet, er essentiel, da det er relateret til, hvor
generelle sådanne systemer er.


Nylig introducerede vi en klasse af ikke-bevarende systemer, der indeholder alle de ovenfor nævnte egenskaber [21, 22, 23]. Arbejdet, der er foregået i samarbejde med Dr. Zeev Olami og Senior scientist Per Bak, Brookhaven National Laboratory, Upton, New York, U.S.A. samt Hans Jacob S. Feder, Universitet i Oslo, Blindern, Norge, er baseret på en simpel model for den dynamiske udvikling af jordskælv. Denne nye klasse af modeller bibringer da også en hærskare af fysiske egenskaber, der synes at være nært knyttet til de turbulente egenskaber ved jordskælv.

Disse ikke-bevarende dynamiske modeller, er de første eksempler på sådanne, der udviser selv-organiseret kritikalitet. Tæthedsfunktionen for den frigivne energi under et jordskælv er en potenslov, hvor eksponenten er en funktion af bevarelsesniveaet, se Figur 5.2. Det kan være den længe søgte forklaring på, hvorfor der observeres forskellige Gutenberg-Richter love for forskellige faults.

Vi har endvidere betragtet den tidslige korrelation for forekomsten af jordskælv og vist, at modellen besidder de samme egenskaber, som observeret for “rigtige” jordskælv: de små jordskælv er ukorrelerede medens de store jordskælv optræder i tidslige klynger, se Figur 5.9.

Modellen indeholder desuden faseovergange, som er irrelevante mht. beskrivelsen af jordskælv men kan have relevans for andre fysiske systemer. En faseovergang viser sig, når bevarelsesniveaet nærmer sig nul. Da introduceres en korrelationslængde i systemet, dvs. tæthedsfunktionen er en eksponentielt aftagende funktion, se ligeledes Figur 5.2, hvorfor systemet ikke længere kan kaldes kritisk.

I nærværende afhandling behandles de bevarende modeller såvel som de ikke-bevarende modeller, således vi kan få indsigt i de forskellige mekanismer, der driver systemerne ind i en kritisk attraktor.
I kapitel 2 diskuterer vi BTW modellen i middel-felts approximatio-

nen, der approximerer grænsen \( d \to \infty \), hvor \( d \) betegner dimensionen. I
middel-felts modellen kan vi identificere kilden til den selv-organisende
proces samt til potensloven for lavine størrelsen. Middel-felts teorien for
BTW modellen er ækvivalent med de såkaldte “branching processes” og
can implementeres vha. modeller, hvor nærmest-nabo cellerne vælges til-
fældigt. Modellerne udvikler sig spontant til en stationær tilstand, hvor
tæthedsfunktionen for lavine størrelsen \( s \) er givet ved

\[
P(S = s) \sim s^{1-\tau_s} \cdot \exp\left(-\frac{s}{\xi(\alpha)}\right),
\]

\[
\xi(\alpha) \to \infty \text{ for } \alpha \to 1.
\]

Parameteren \( \alpha \) er relateret til bevarelsesniveauet i modellen. Hvis
\( \alpha = 1 \) er modellen bevarende, og systemet er kritisk, idet tæthedsfunktionen
udelukkende er en potenslov. Enhver form for ikke-bevarelse \( (\alpha < 1) \) in-
troucerer en endelig korrelationslængde \( \xi \). Vi viser, at adskillige mod-
eller tilhører samme universalitets klasse som “branching processes”.

Endelig gør vi rede for, at middel-felts beskrivelsen af skovbrands
modellen (eng. forest-fire model), introduceret af B. Drossel og F. Schw-
abl [35], er identisk med “branching process” beskrivelsen. Det medfører,
at tæthedsfunktionen for antallet af nedbrændte træer i en enkelt skovbrand
er en potenslov med eksponenten \( \tau_s = 5/2 \). Foreløbige simulationer af
skovbrands modellen viser, at eksponenten nærmer sig middel-felts værdi-
dien, når dimensionen øges, hvilket strider imod hypotesen, at eksponen-
tet \( \tau_s = 2 \) uanset dimensionen.

I kapitel 3 introducerer vi BTW modellen på et gitter, der definerer
nabo relationerne. Dermed vil der introduceres rumlige korrelationer i
modellen samtidig med, at problemet kan formuleres som et transport
problem, jvf. den indledende diskussion vedrørende sandbunken. I et
Bethe gitter vil BTW modellen give anledning til eksponentielt aftagende
korrelationer medens et hyperkubisk gitter introducerer korrelationer,
der aftager som potenslove.

Endvidere vil middelværdien af lavine størrelsen scale med system
størrelsen. Dette hidrører fra, at strømmen af sandskorn ind i sys-
temet nødvendigvis må transporteres til randen af systemet, hvor sand-
skornene kan dissiperes. Som nævnt er tæthedsfunktionen for lavine
størrelsen en potenslov. Vi giver heuristiske argumenter, som igen er
relaterede til transport egenskaberne ved modellen, der viser kilden til
denne potenslov.
Endelig udleder vi scalings relationer mellem de kritiske eksponenter. Yderligere band på de kritiske eksponenter kan udlede, så vi alt i alt har \( N - 1 \) band på \( N \) kritiske eksponenter. Vi bestemmer én af eksponenterne via en simulering, hvorved vi kan fastlægge de resterende. Eksempelvis finder vi, at eksponenten \( \tau_s \), der beskriver lavine størrelsen, er givet ved \( 48/23 \approx 2.09 \), sammenlignet med den målte værdi på 2.08(5).


Vi får et tydeligt billede af støjens rolle, når vi betragter ikke-bevarende systemer med lokale perturbationer. En sådan model udviser en faseovergang fra et ikke-kritisk til et kritisk system, når støj, der genereres af perturbationerne selv, bliver mindre end en endelig størrelse.


Man havde en formodning om, at de kritiske BTW modellerne måske kunne udgøre en mulig forklaring på fænomenet \( 1/f \) støj. Men fraværet af
indre længde skalaer i såvel rum som tid udmønter sig ikke nødvendigvis i ikke-trivielle eksponenter i power spektret. Det viste sig snart, at power spektret for BWT modeller er $1/f^2$ [41, 59, 60]. I kapitel 6 ser vi, at $1/f$ støj alligevel kan forklares som et deterministisk, selv-organiseret kritisk fænomen, der opstår spontant i ikke-bevarende systemer med mange vekselvirkende frihedsgrader [24].
Bibliography


[37] In the language of Dhar, a model is undirected if the toppling matrix $\Delta$ is symmetric.


