PERCOLATION

One of the most direct ways of understanding the idea of a continuouss phase transition is to study percolation [4]. It is a purely geometrical phase transition and requires no prior understanding of statistical mechanics (although percolation may be viewed statistical mechanically [2]). For simplicity, think of a two-dimensional regular square lattice of size $L \times L$. Each site may be either occupied, with probability p, or unoccupied, with probability $(1-p)$. The state of each site is chosen independently of all other sites. In the percolation animation, occupied sites are coloured yellow, while unoccupied sites are coloured blue.

The fundamental object of interest in percolation is a cluster, taken to be a collection of occupied sites that are 'connected' via their nearest-neighbours. It is intuitively clear that when the occupation probability p is small, the clusters that form by chance are likely to be small in size. Conversely, when p is large (i.e., close to one), clusters are likely to be large. The occupation probability may be changed in the animation by typing in a new value in the relevant field. As we have learned, there will be a whole spectrum of clusters sizes for any given realisation. We are interested in the cluster number density $n(s,p)$, for a cluster of size s at occupation probability p. While it is difficult to confirm the form of $n(s,p)$ visually, a crude indication of the range of cluster sizes present in a given realisation may be obtained by considering the largest cluster. In the percolation animation, the largest cluster is coloured red, and its size, in terms of the number of occupied sites, is indicated on the left. As expected, the larger the value of p, the larger the largest cluster size. Loosely speaking, the largest cluster size (averaged, say, over a large number of realisations) may be thought of as a characteristic cluster size s_ξ, because clusters larger than this size are very rare.[1]

Percolation becomes most interesting near the critical occupation probability p_c. Recall that the order parameter for percolation is the density of the infinite cluster. Of course, we are only able to simulate finite systems, so we will never encounter an infinite cluster. Let us therefore think of the largest cluster instead. For two-dimensional percolation on a square lattice, the critical occupation probability is not known exactly, but numerical studies have closed in on $p_c = 0.592\,746\,21\ldots$[3]. Just above this value of p_c, the density of the largest cluster increases dramatically, and this may be confirmed visually in the animation. The nature of the geometry of the largest cluster at the critical occupation probability p_c will no doubt be recognisable to many of you. It is an example of a fractal. The best way to view it is by applying the 'isolate' toggle, which will remove all other clusters. The largest cluster is a rarefied object: it is perforated by holes of all sizes (or, at least, as large as the system size).

[1] Unfortunately, the largest cluster size is not quite analogous to the characteristic cluster size $s_\xi(p)$ defined in the book, since the largest cluster does in fact depend not only on the occupation probability p, but also on the system size L (the dependence is logarithmic [1]).

It has a fractal dimension of $D = 91/48 \approx 1.90$. Since this is less than than $d = 2$, the dimension of the system, the density of the largest cluster actually decreases with increasing system size! This is one of the many curious features of a fractal. In practice, the largest cluster may or may not percolate the system. That is, it may or may not extend from one boundary to the other opposite boundary. But if the occupation probability p is increased a little above p_c, the largest cluster percolates virtually all the time, and the room left for 'finite' (non-percolating) clusters rapidly diminishes.

References

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